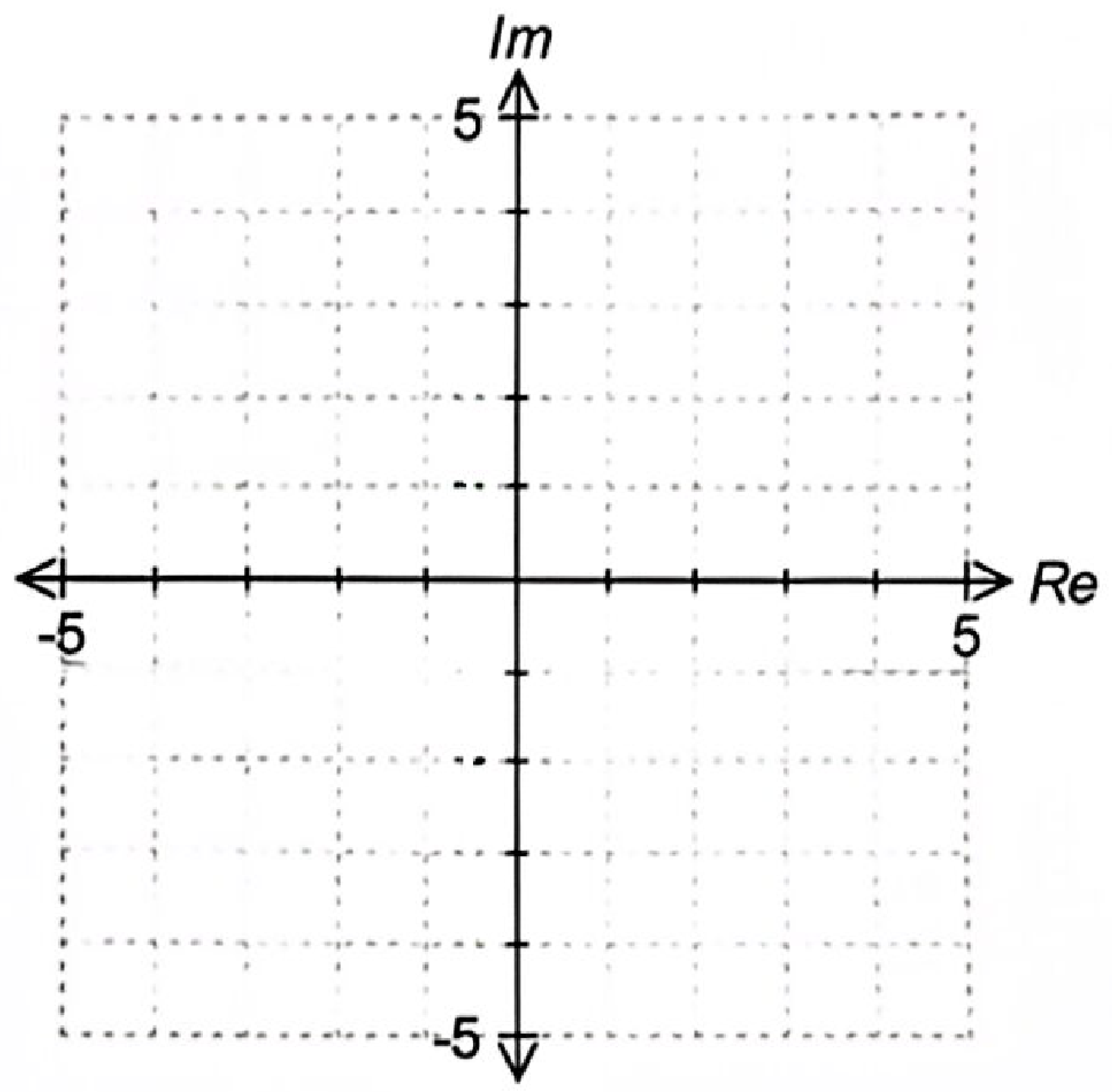
02 Complex Numbers II

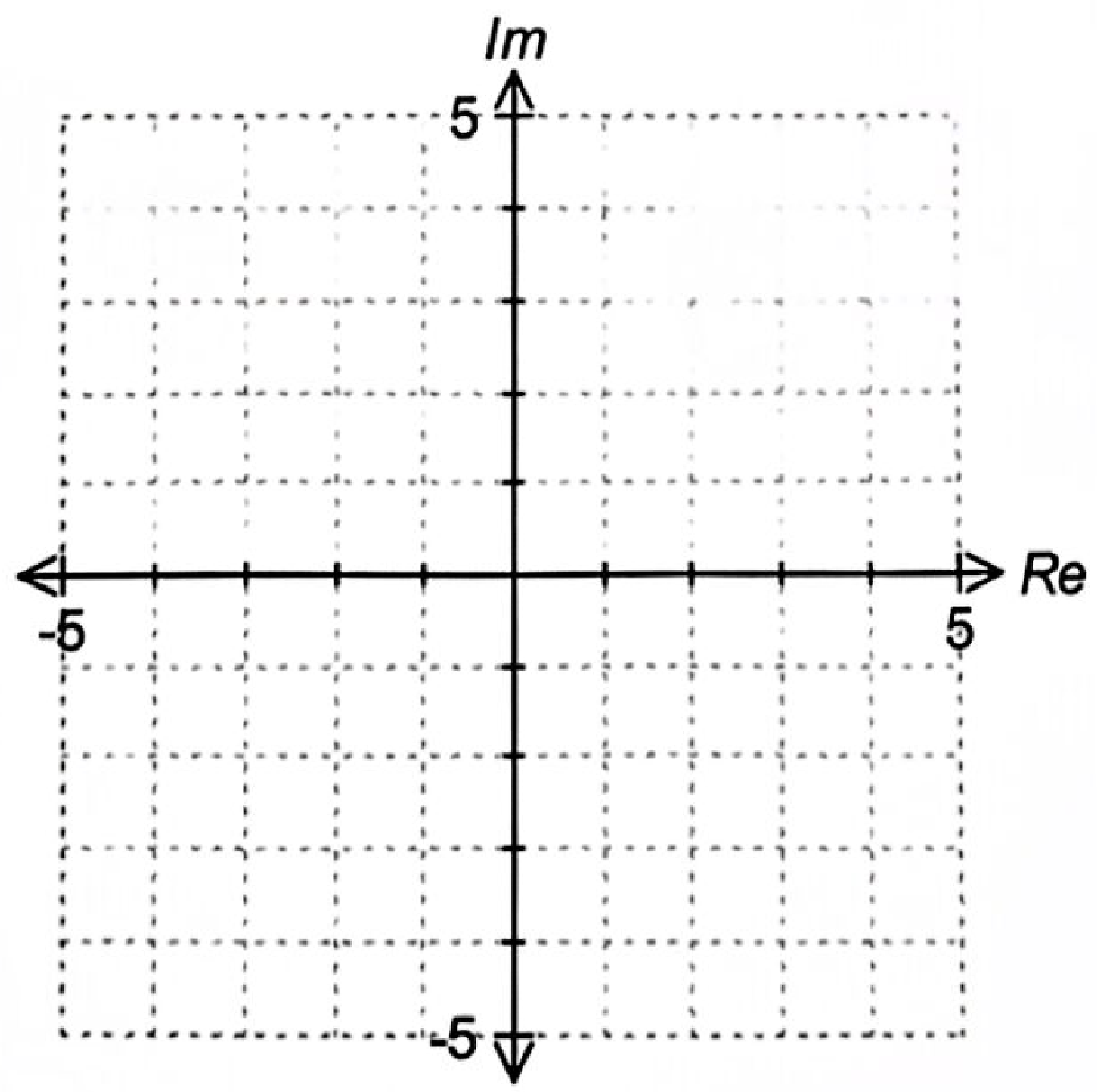
Calculator Free

1. [13 marks: 2, 2, 2, 3, 4] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{z : \text{Im}(z) \geq |\text{Re}(z) + 1|\}$.



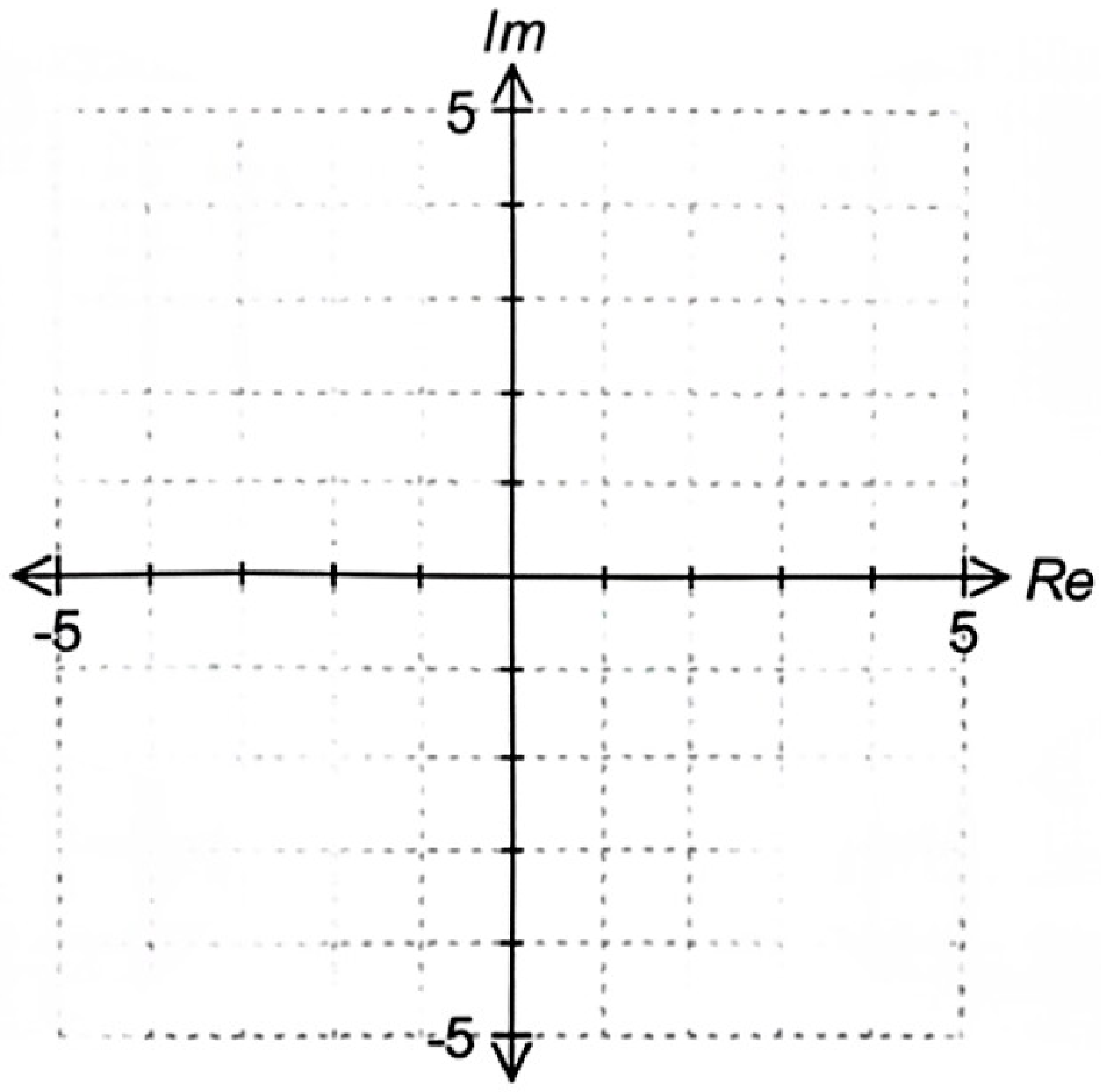
(b) Sketch the region in the Argand Plane defined by $\{ z : |z - 1| \geq |z + 1 - 2i| \}$



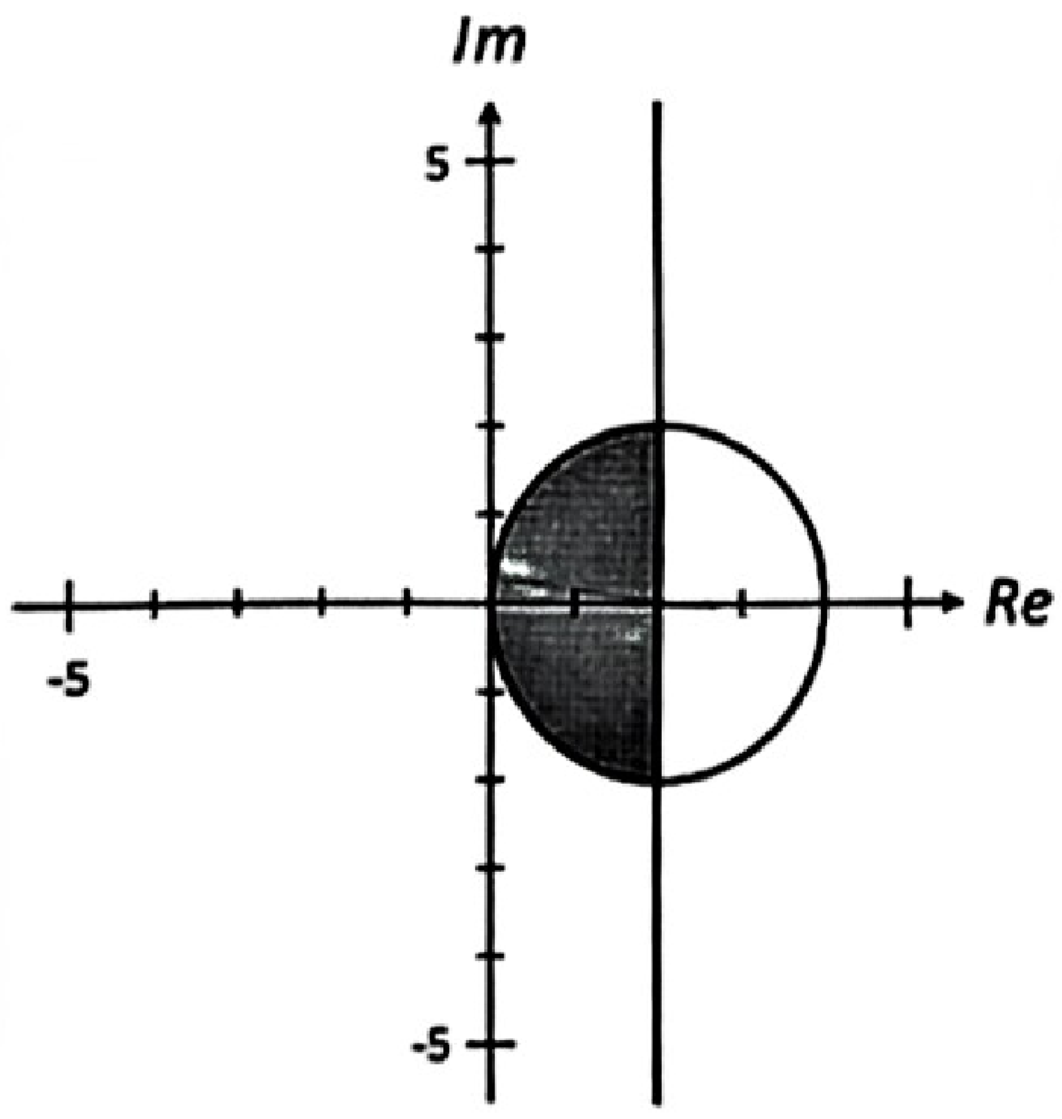
Calculator Free

1. (c) (i) Let $z = r\ cis\ \theta$ where $0 < \theta \leq \pi$. Show that $\text{Arg}(z^2) = 2\theta$.

(ii) Sketch the region in the Argand Plane defined by $\{z : \text{Arg}(z^2) \geq \frac{\pi}{2}\}$



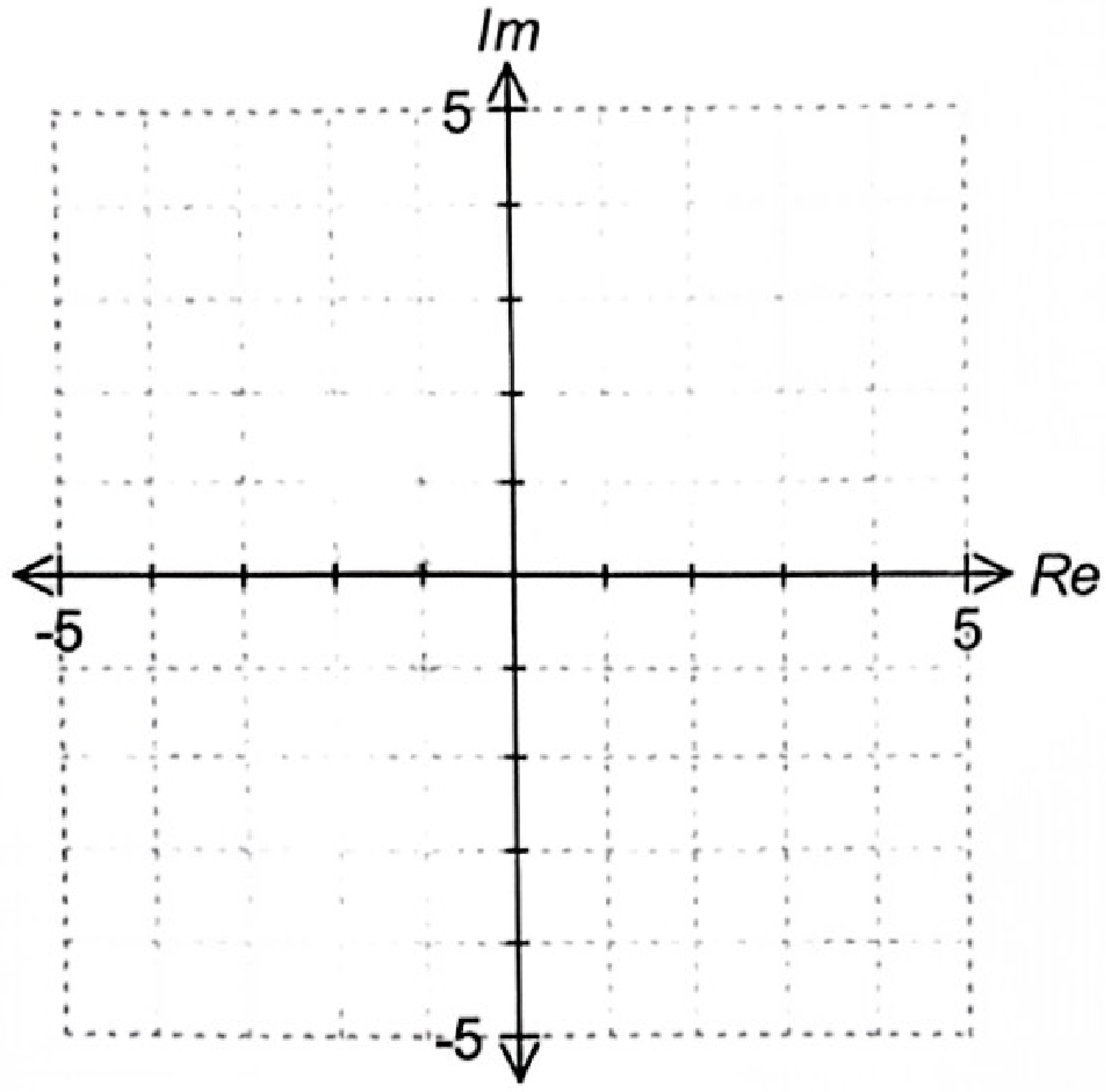
(d) Describe the complex set that defines the region in the Argand Plane shown below.



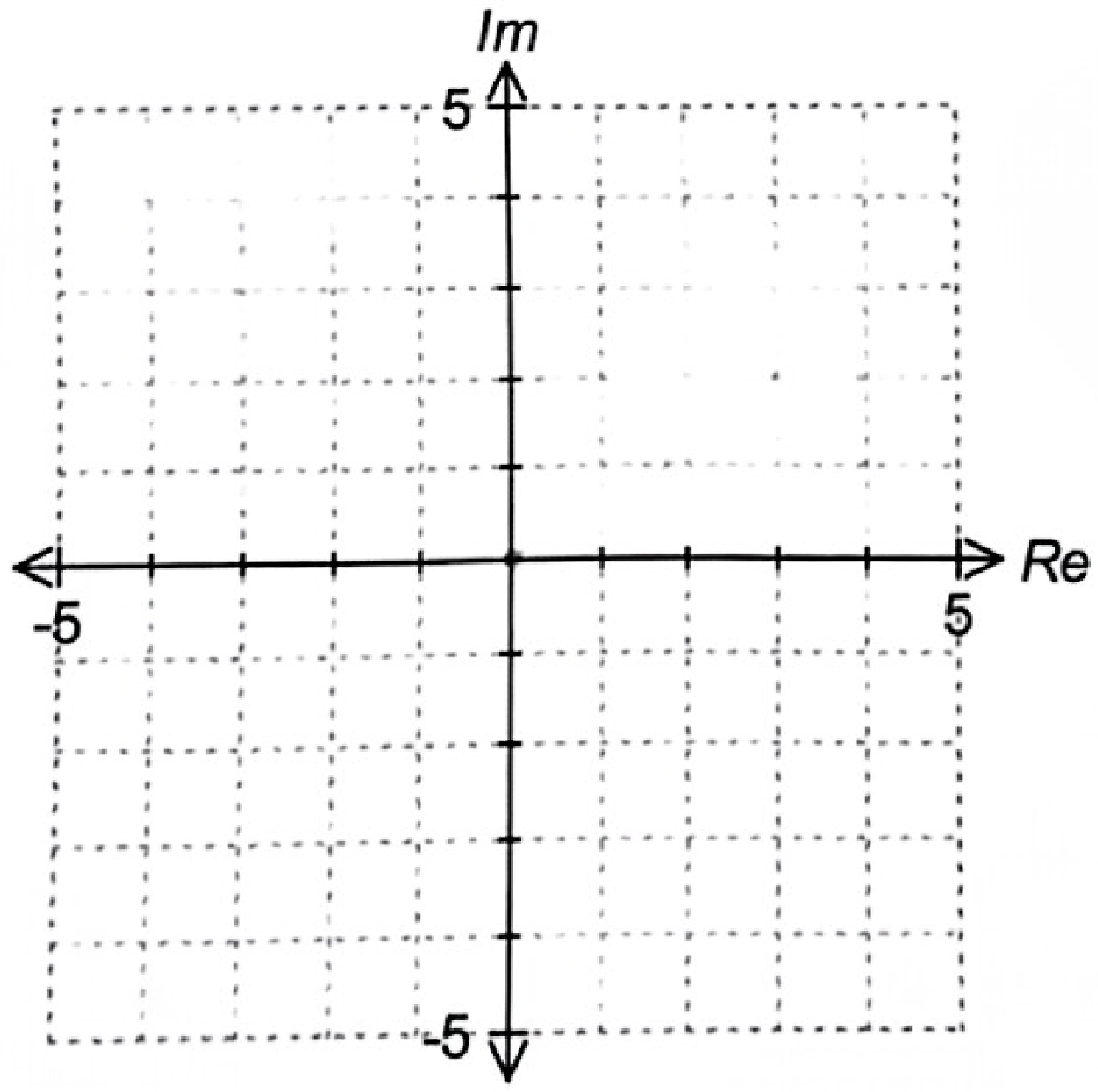
Calculator Free

2. [8 marks: 2, 2, 2, 2] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{z : |z + i| = 1\}$.

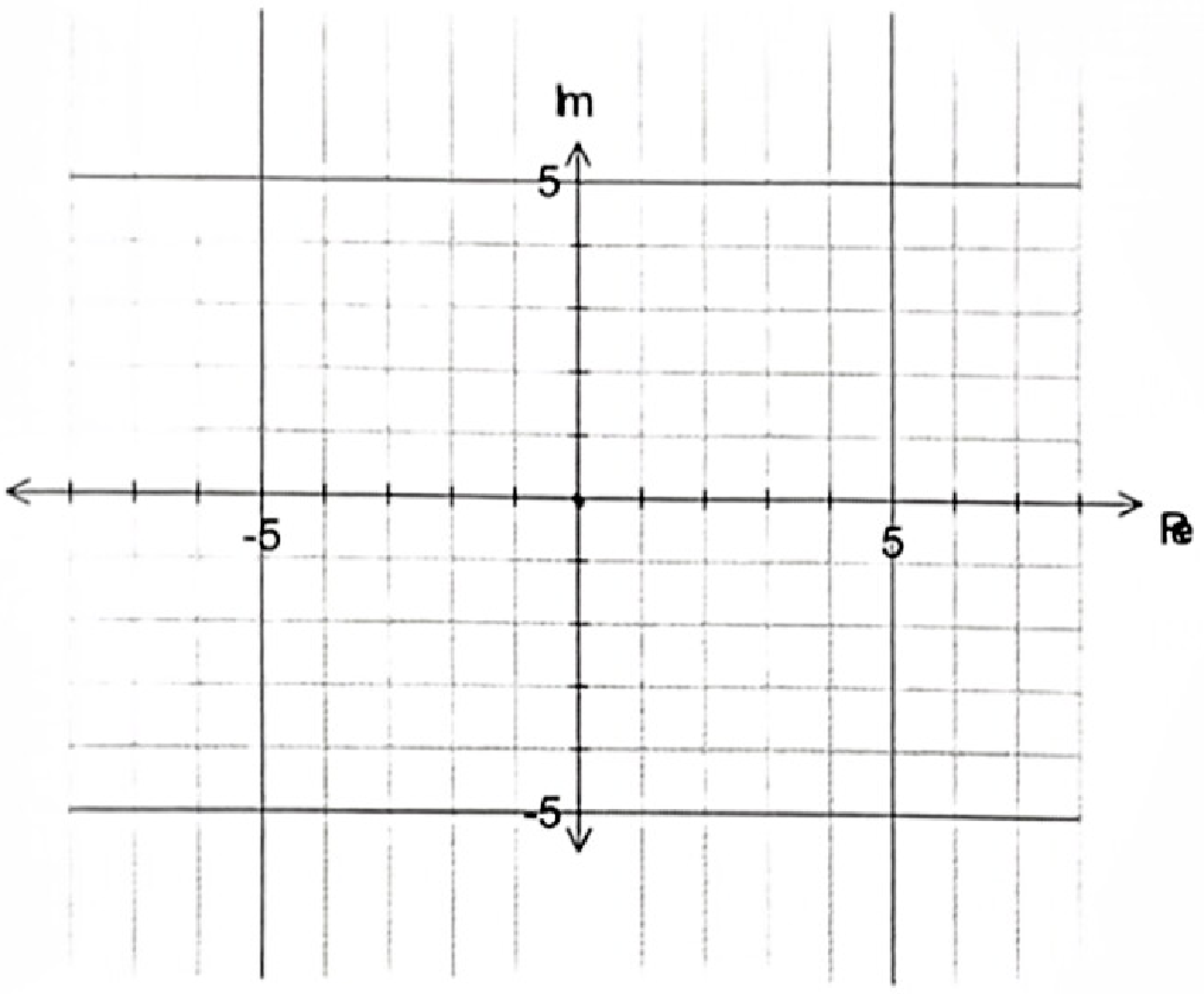


(b) Sketch the region in the Argand Plane defined by $\{ z : arg\,(z) = \frac{\pi}{4}\}$.

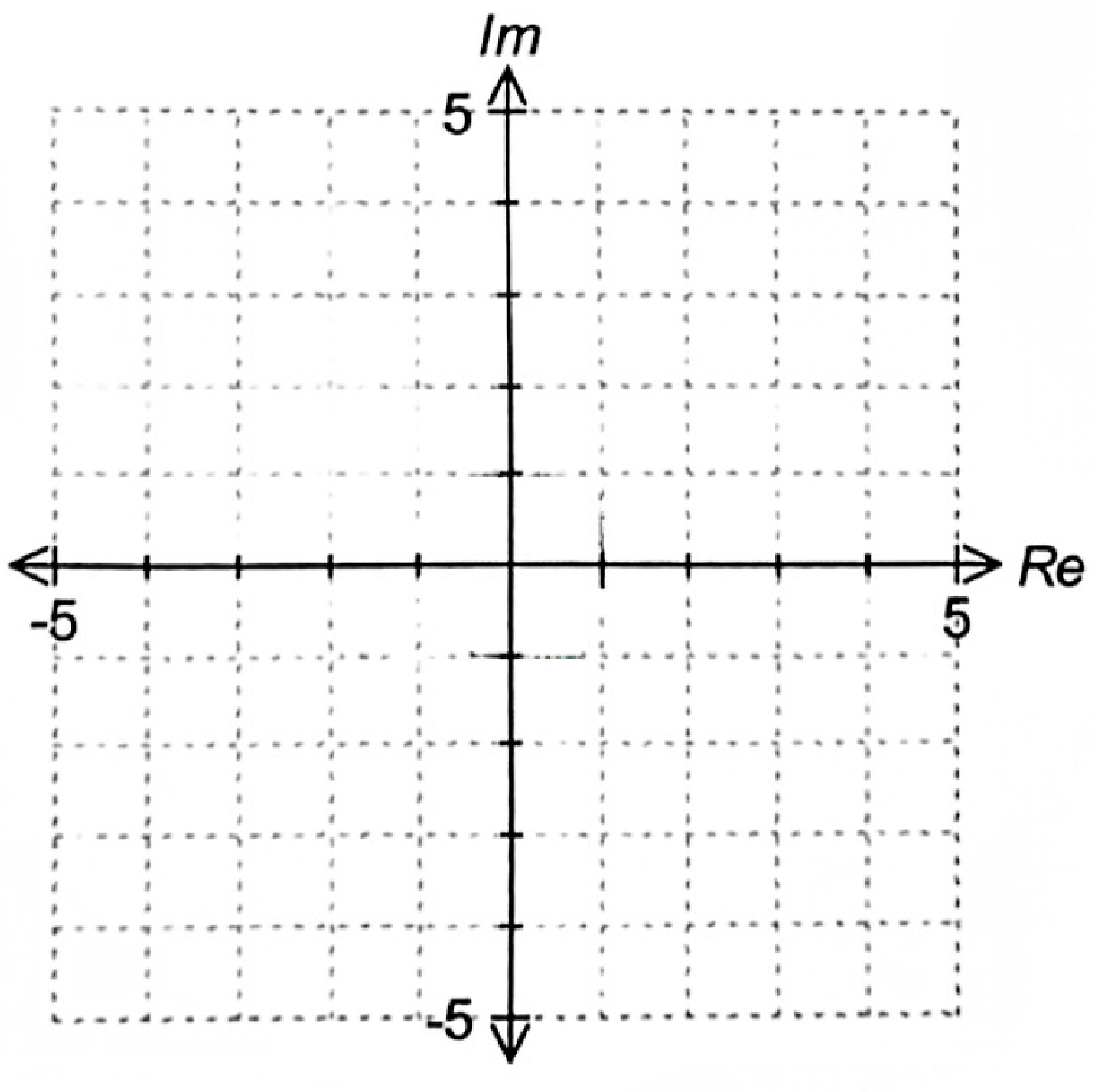


Calculator Free

2. (c) Sketch the region in the Argand Plane defined by
$\{ z : |z| = arg(z) \text{ where } 0 \le arg(z) \le 2\pi \}$.



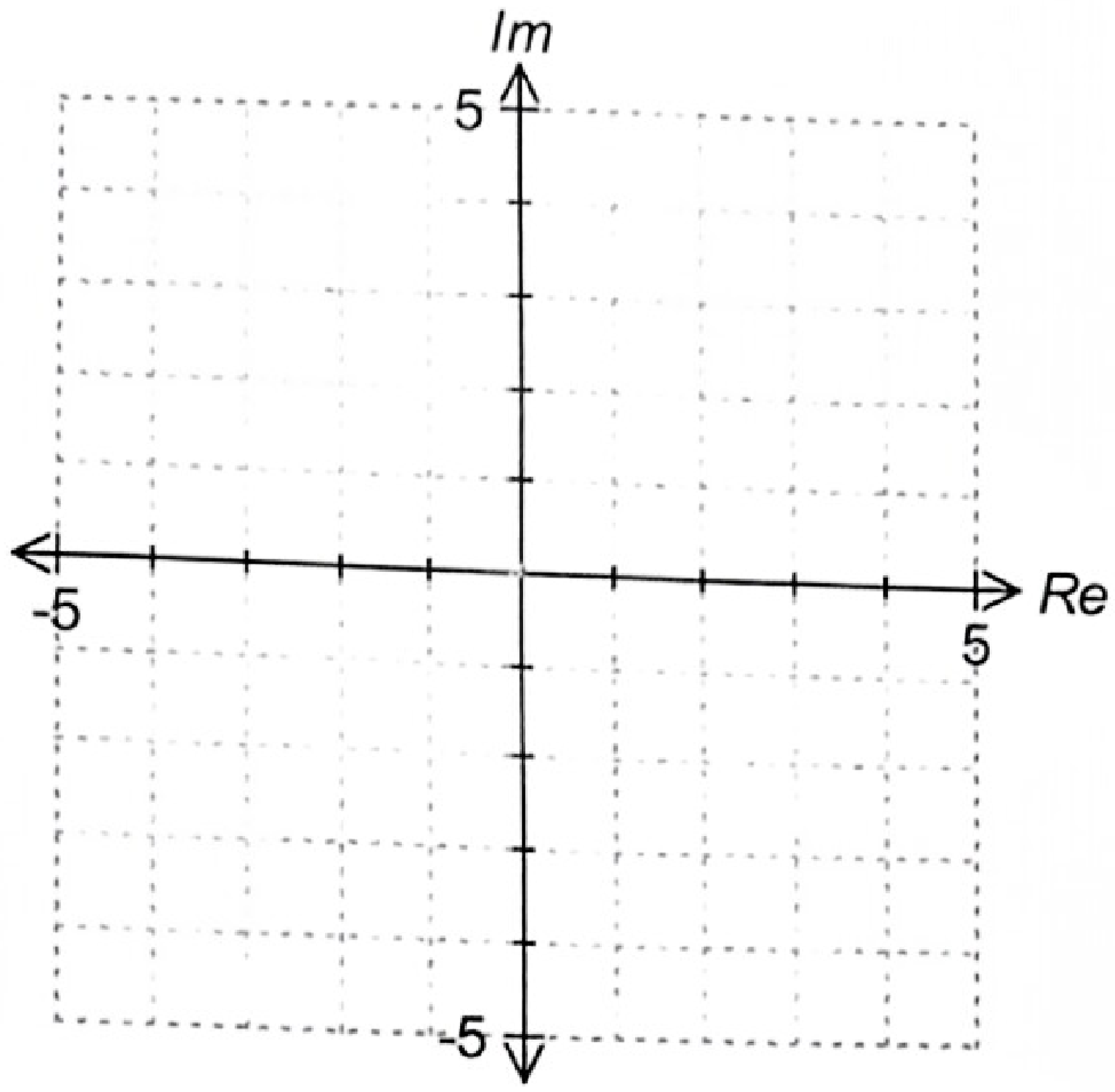
(d) Sketch the region in the Argand Plane defined by
$\{z : z = \cos\theta + i\sin\theta \text{ where } -\pi < \theta \le \pi\}$.



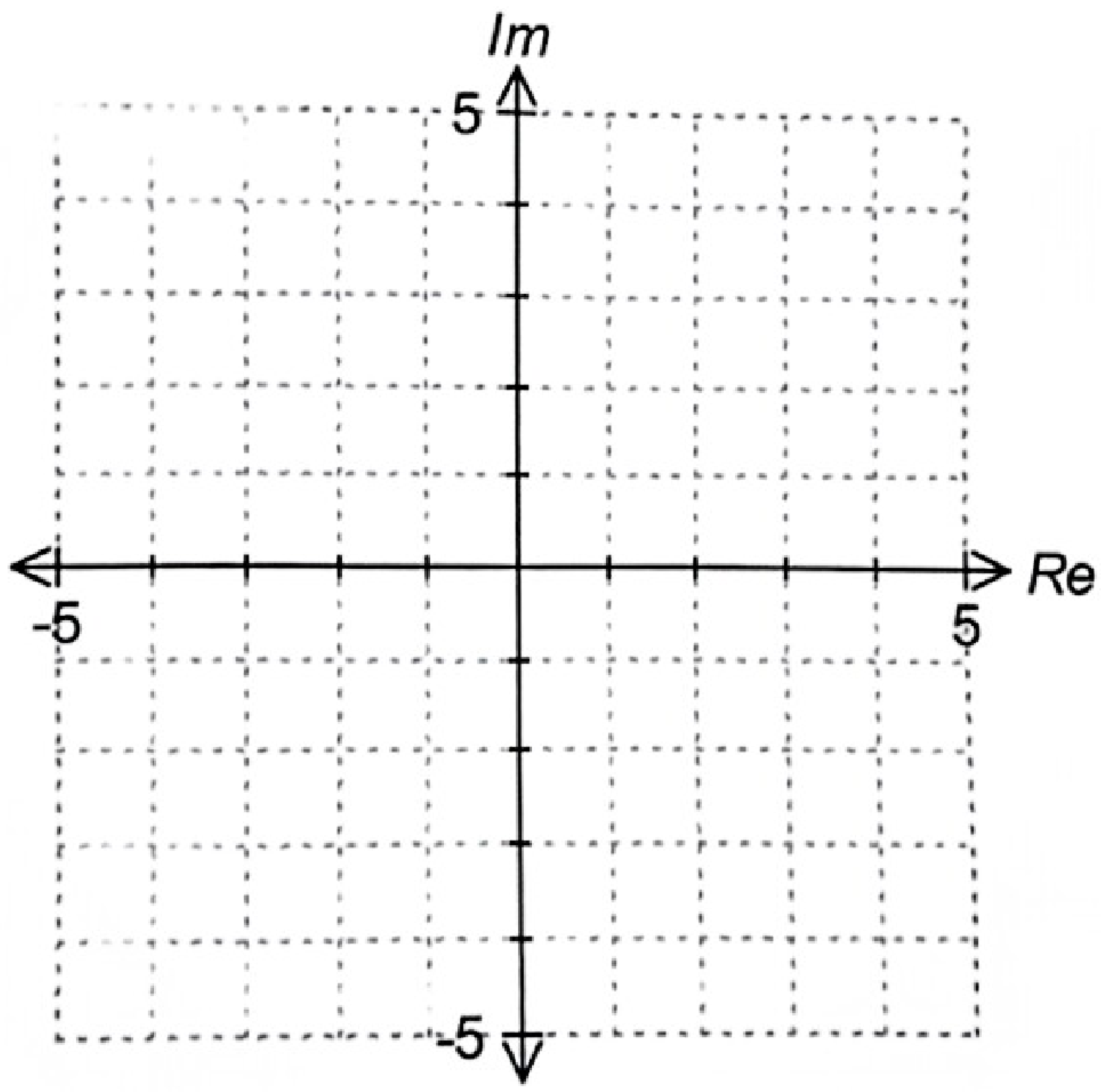
Calculator Free

3. [10 marks: 2, 2, 3, 3] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{ z : |z| = \frac{\pi}{4} \}$.



(b) Sketch the region in the Argand Plane defined by $\{z : \tan [arg(z)] = 1\}$.



Calculator Free

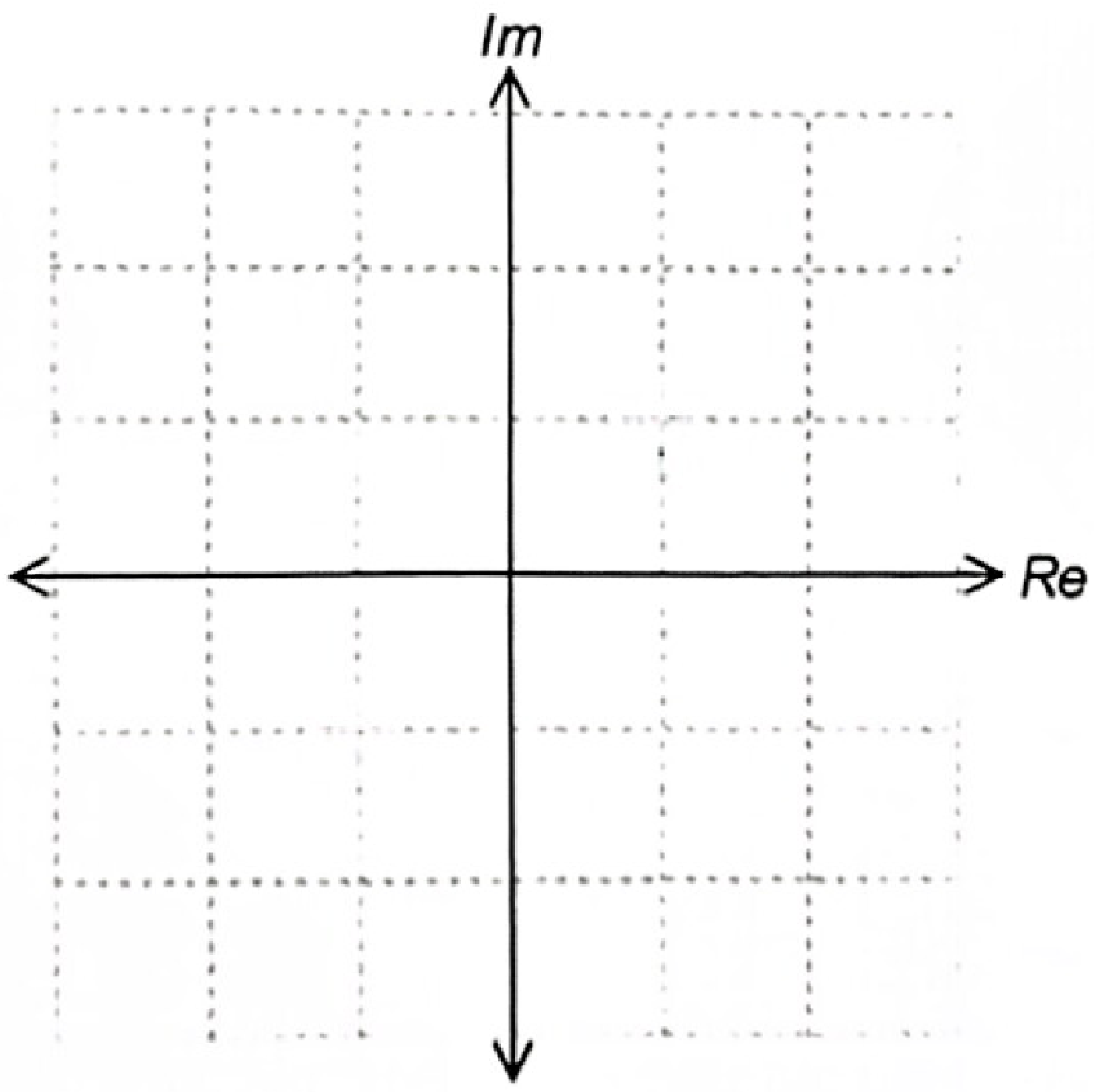
3. (c) Consider the region in the Argand Plane defined by $\{z : z^2 = 2i\}$.

Let $z = x + iy$ where x and y are real numbers.

(i) Show that the Cartesian equation of this region is given by $x^4 = 1$.

(ii) Hence, show that this region consists of exactly *two* points.

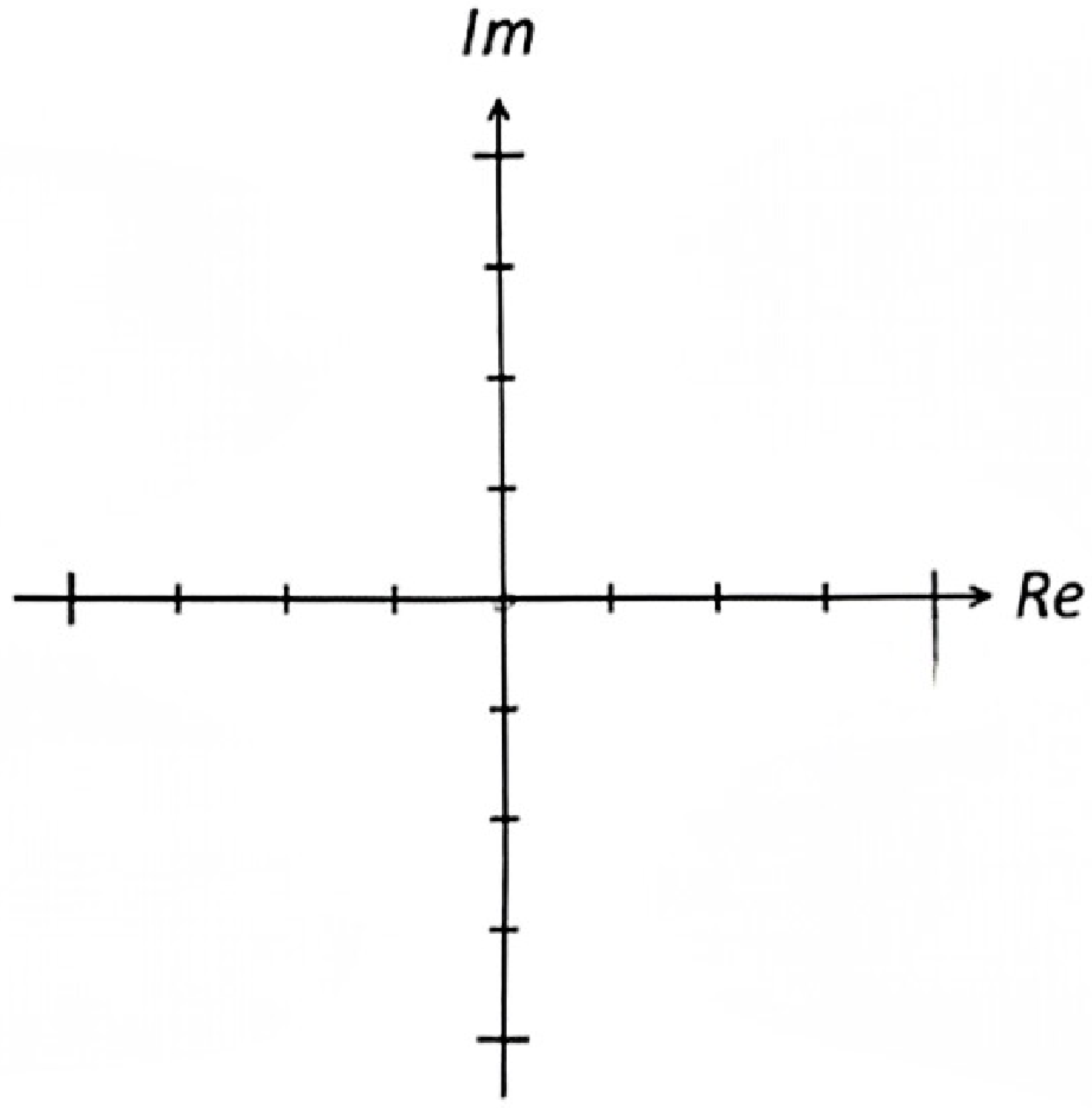
Mark these two points clearly on the axes below.



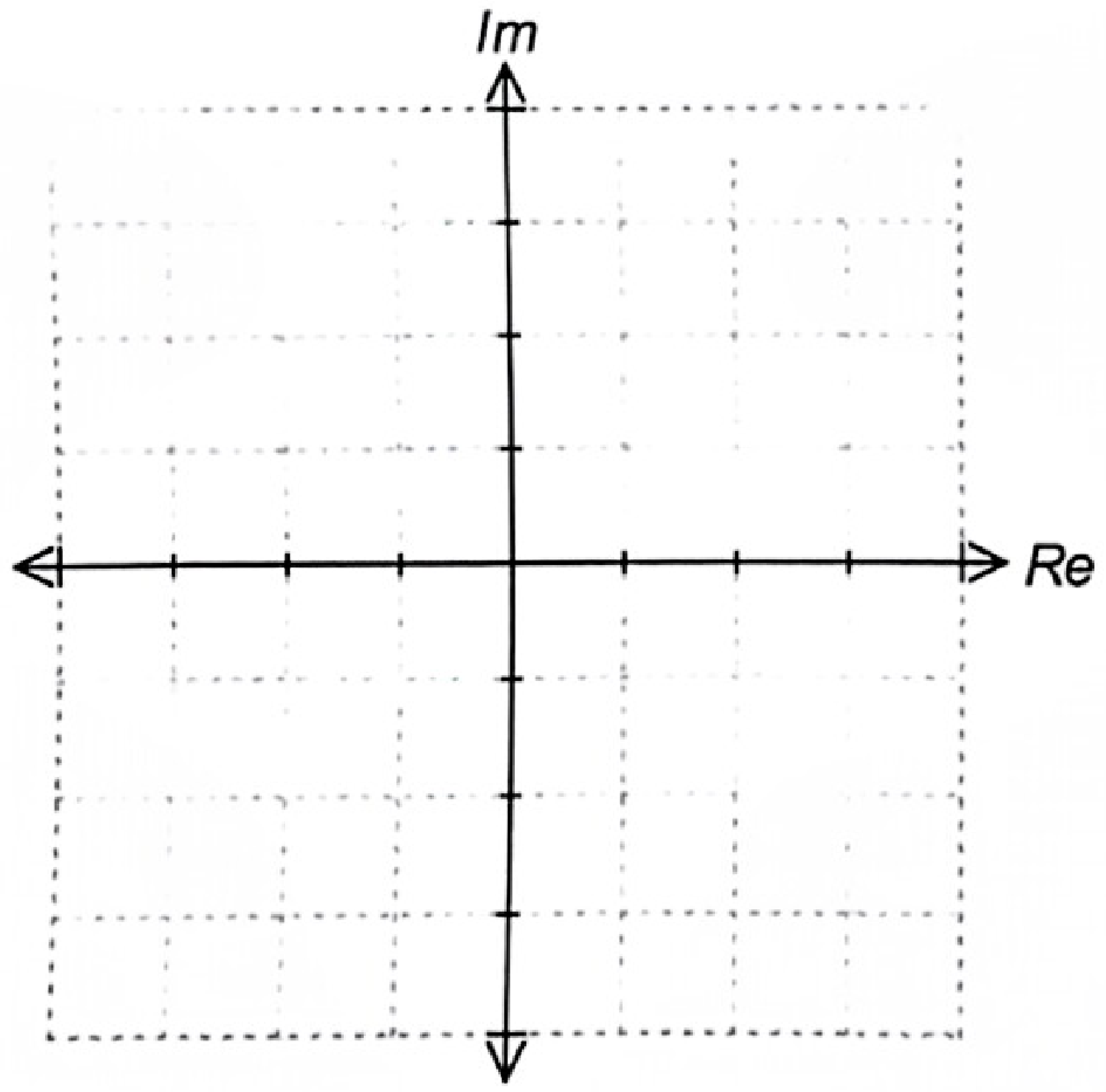
Calculator Free

4. [10 marks: 2, 2, 3, 3] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{ z : |z| = \frac{\pi}{2} \}$.

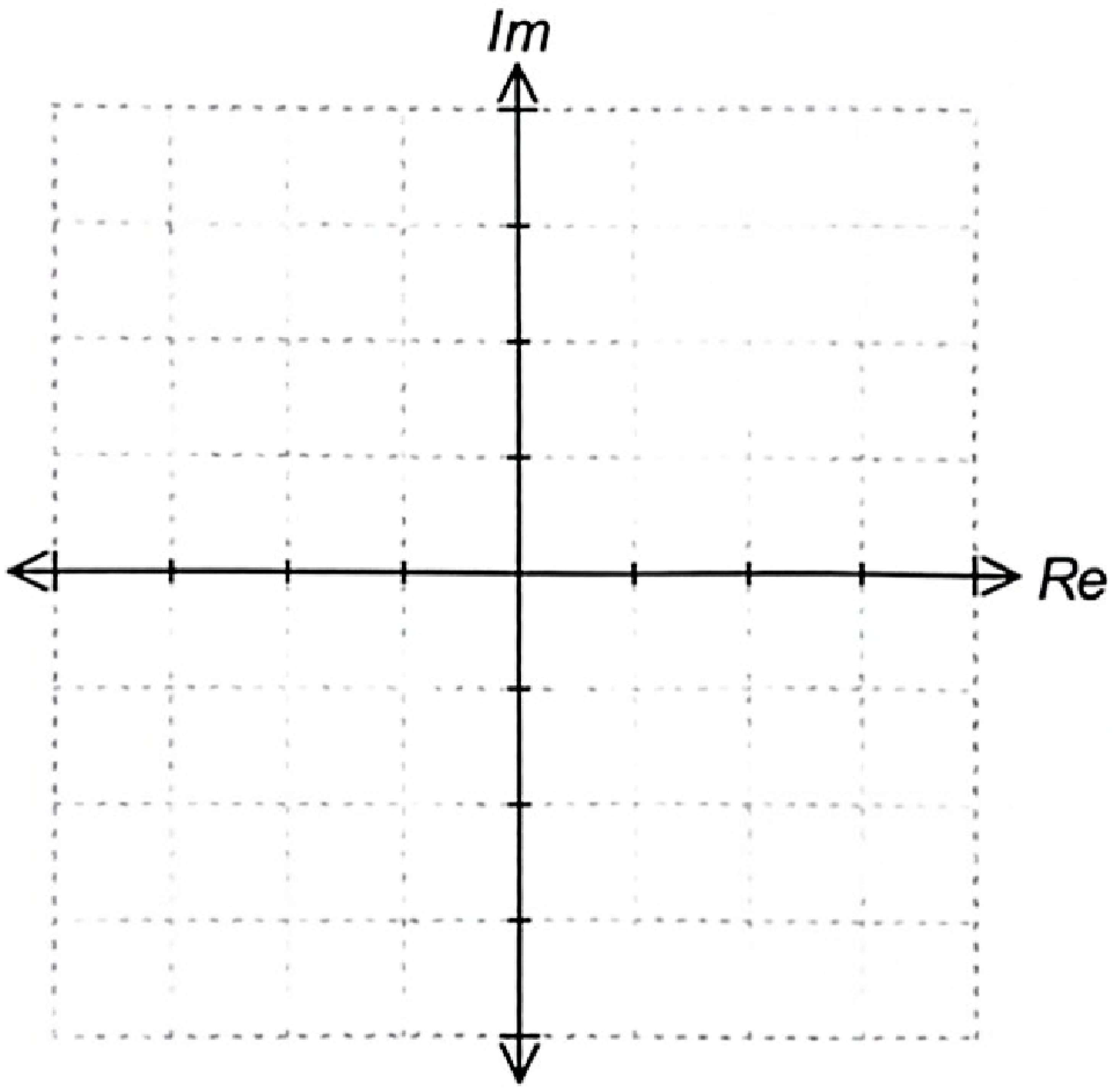


(b) Sketch the region in the Argand Plane defined by $\{ z : |arg(z)| = \frac{\pi}{4} \}$.



Calculator Free

4. (c) Sketch on the diagram below the locus of the point z defined by:
$\{z : |z + 3 + 2i| + |z - 4 + 2i| = 7\}$.

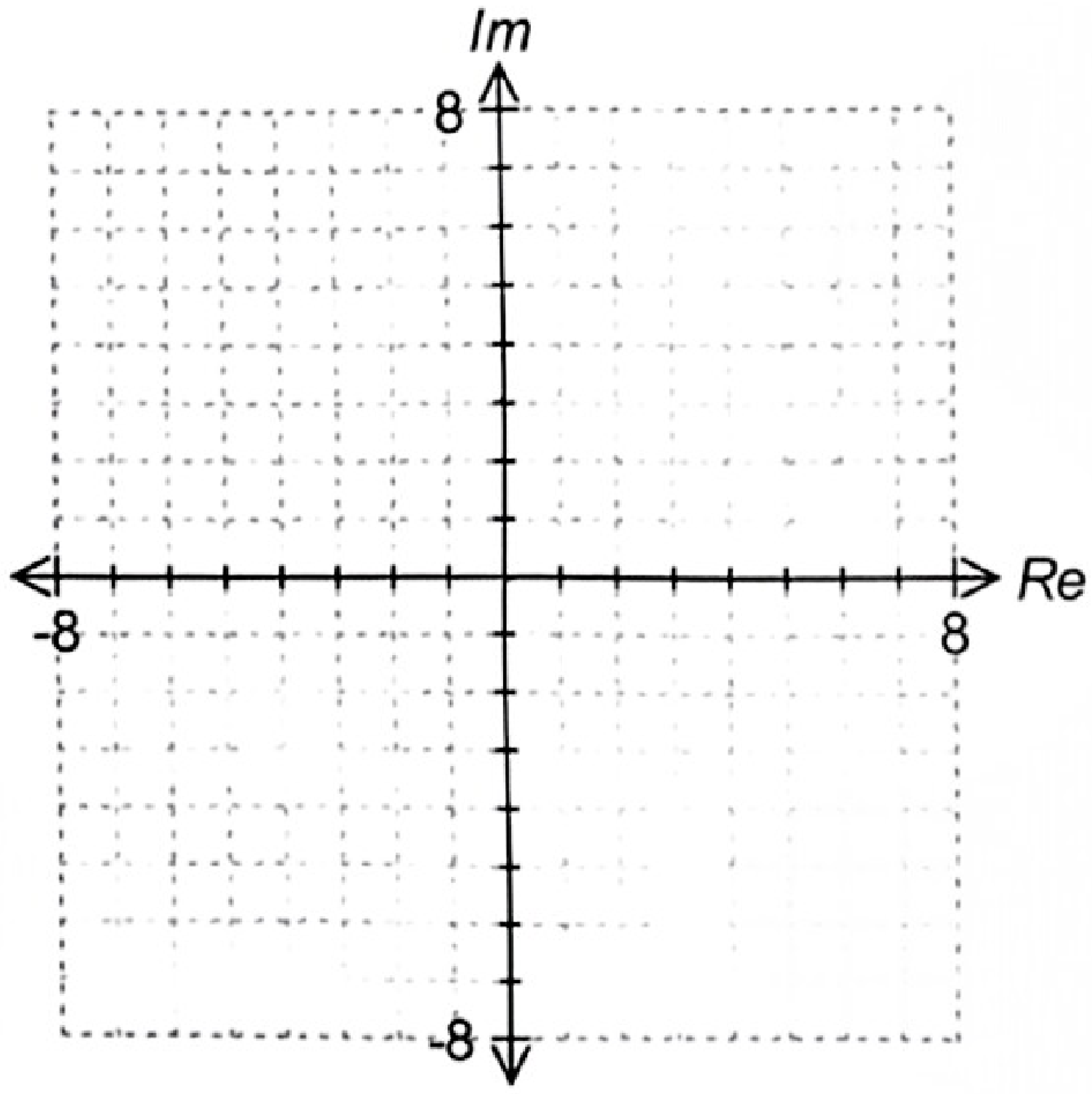


(d) Region R in the Argand Plane is defined by $\{ z : |z - 1| \le |z + i|\}$.
Region R can also be described in Cartesian form by the inequality $ax + by \ge 0$. Find a and b. [Hint: Use a sketch.]

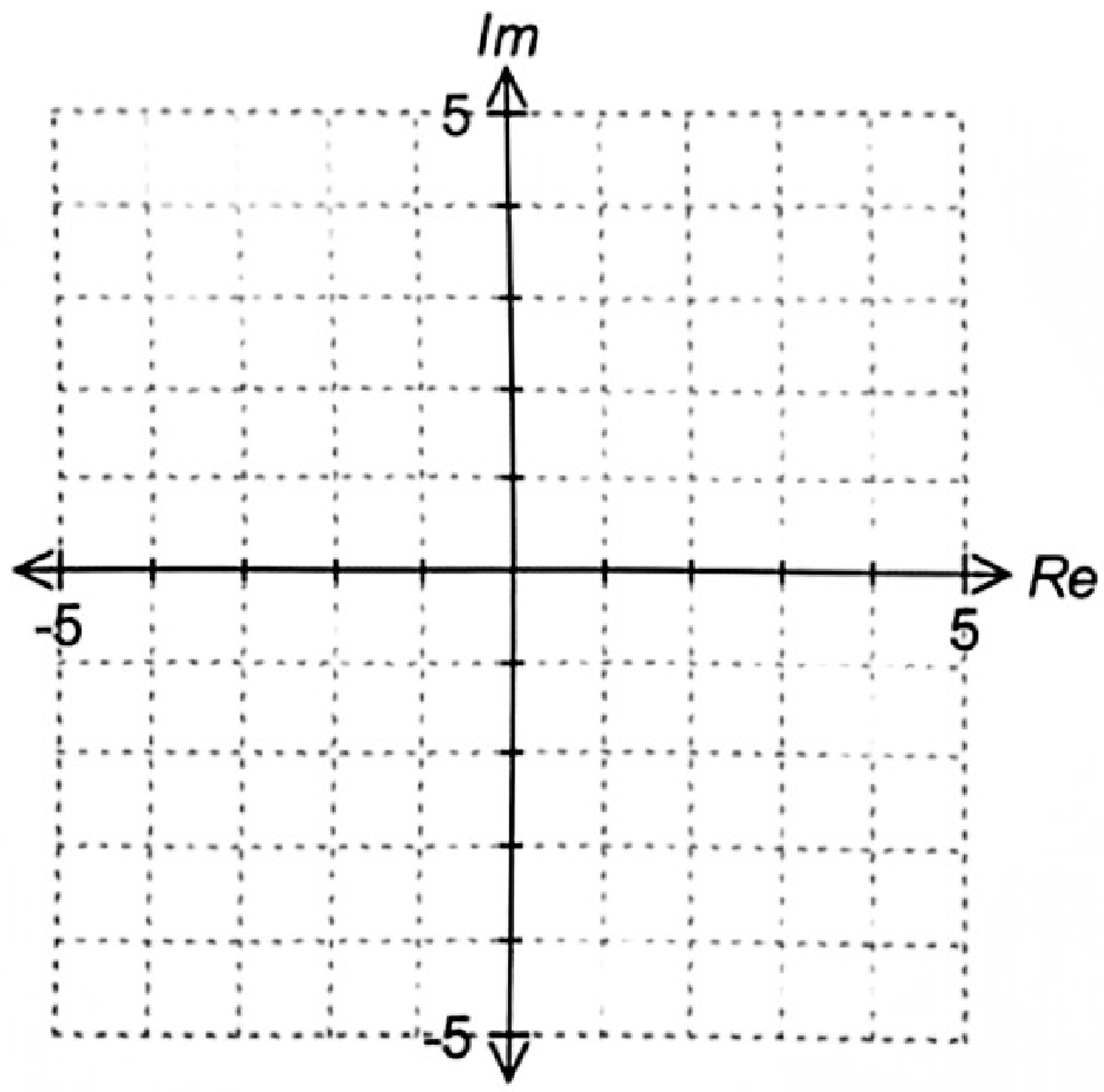
Calculator Free

5. [11 marks: 2, 2, 3, 4] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{ z : |\bar{z}| = 5\}$.



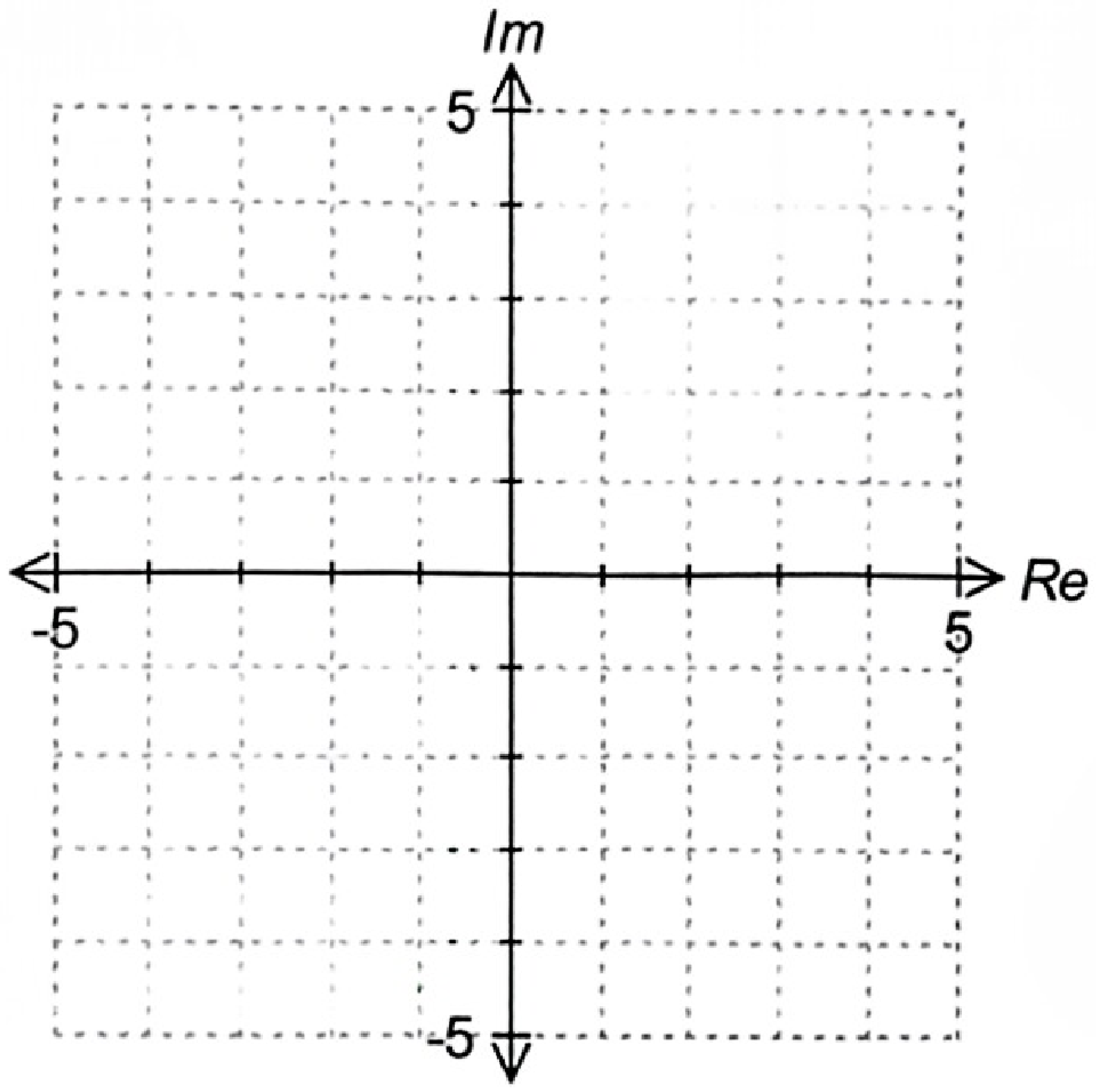
(b) Sketch the region in the Argand Plane defined by $\{ z : arg(\bar{z}) = \frac{-\pi}{4}\}$.



Calculator Free

5. (c) Sketch on the diagram below the locus of the point z defined by:

$\{z : |z + 2 + 2i| \le 2 \cup 0 \le arg\,(z) \le \frac{\pi}{2}\}$.

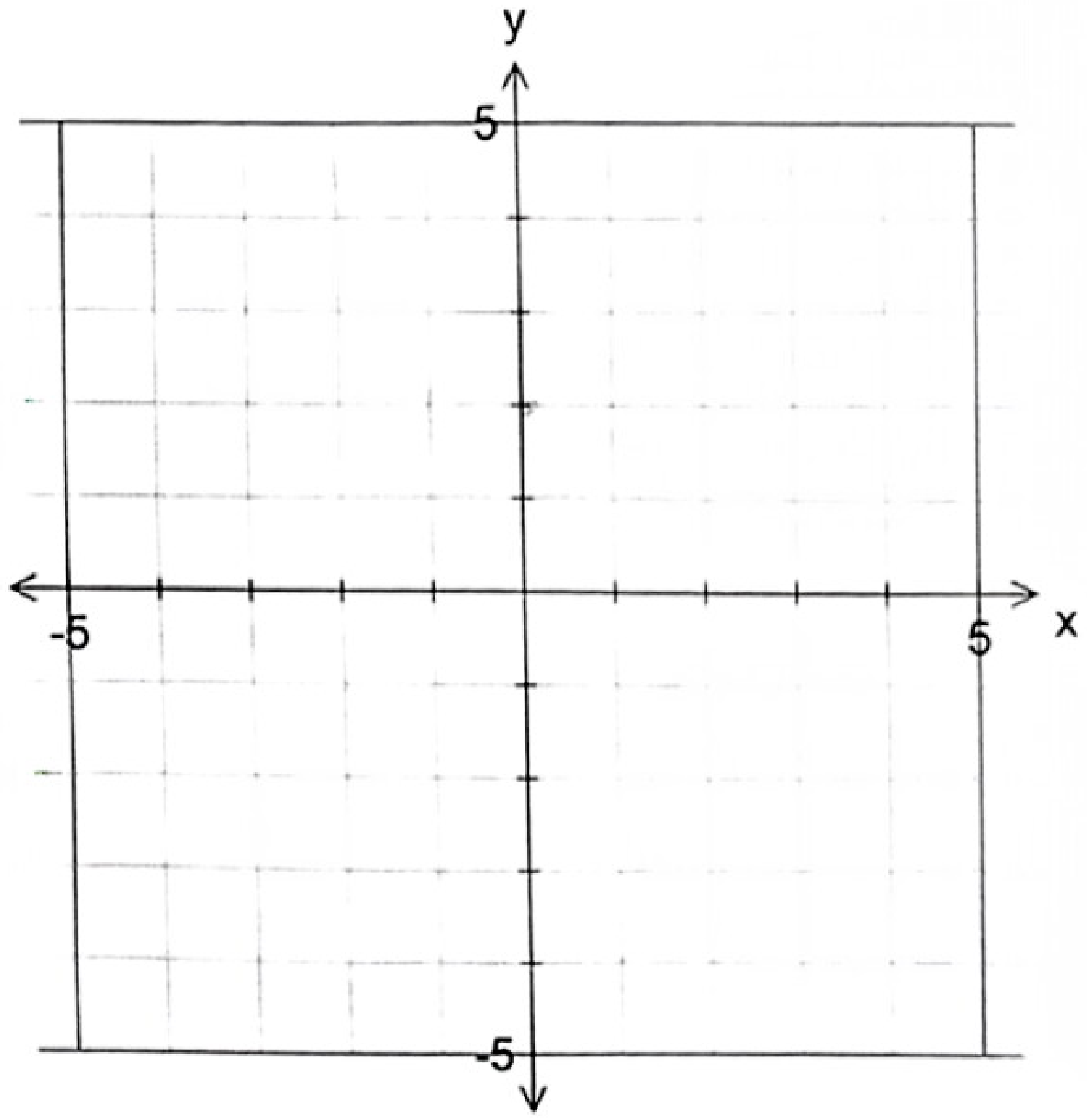


(d) Find, in its simplest form the Cartesian equation of the locus of the point z defined by $|z - 1 - i| = \mathbf{Re}\,(z + 3 + 4i\,)$.

Calculator Free

6. [12 marks: 2, 3, 7] [TISC]

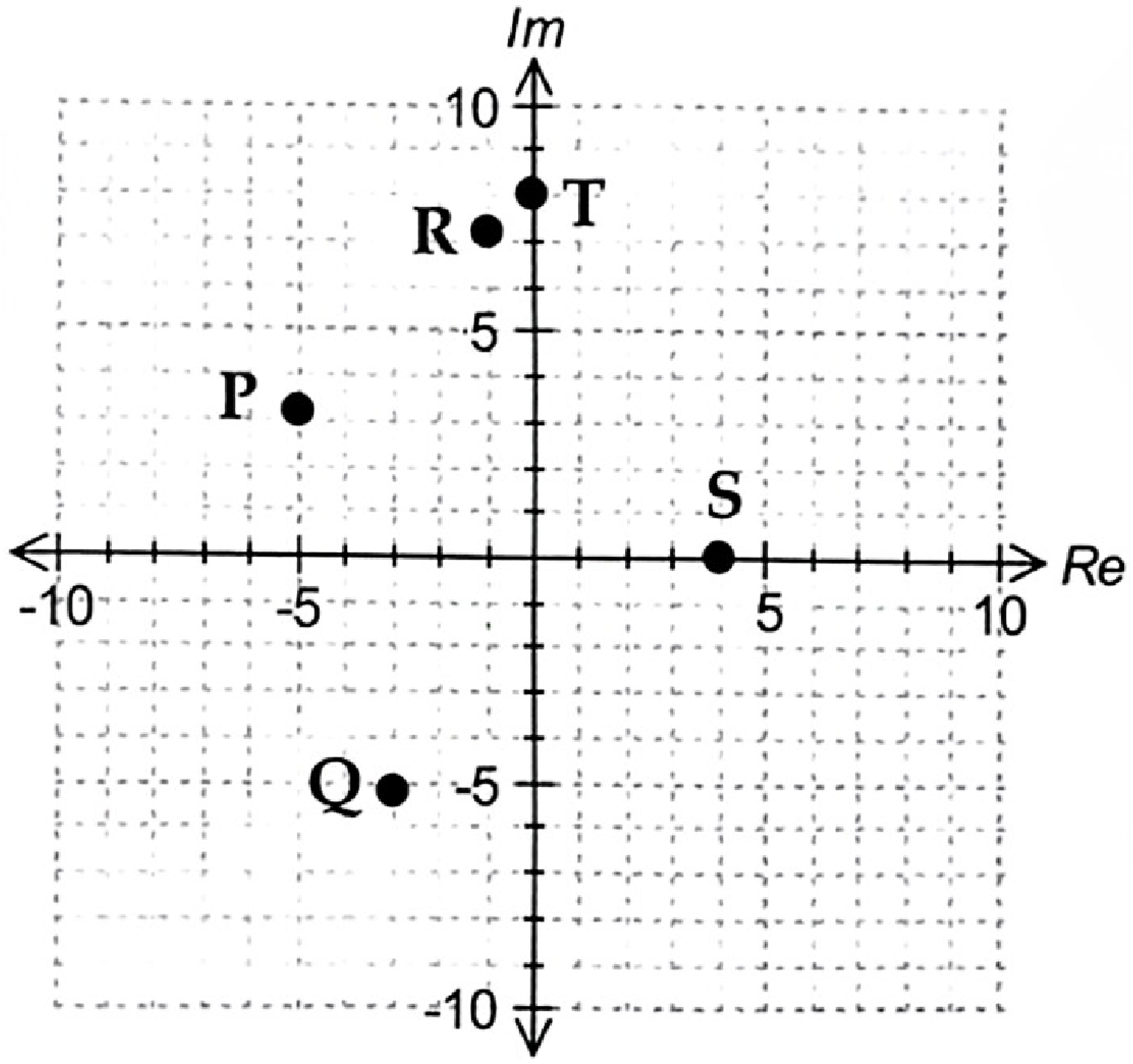
(a) Sketch on the diagram below the locus of the point z defined by:
$\{z : |z - 2i| \geq 1\}$.



(b) Find, in simplest form the Cartesian equation of the locus of the point z defined by $|z - 1| = |z - 1 + 2i|$.

Calculator Free

6. (c) Consider the complex numbers $u = 2 + 2i$ and $v = -3 + 3\sqrt{3}\,i$.
The Argand diagram below shows the points P, Q, R, S and T.

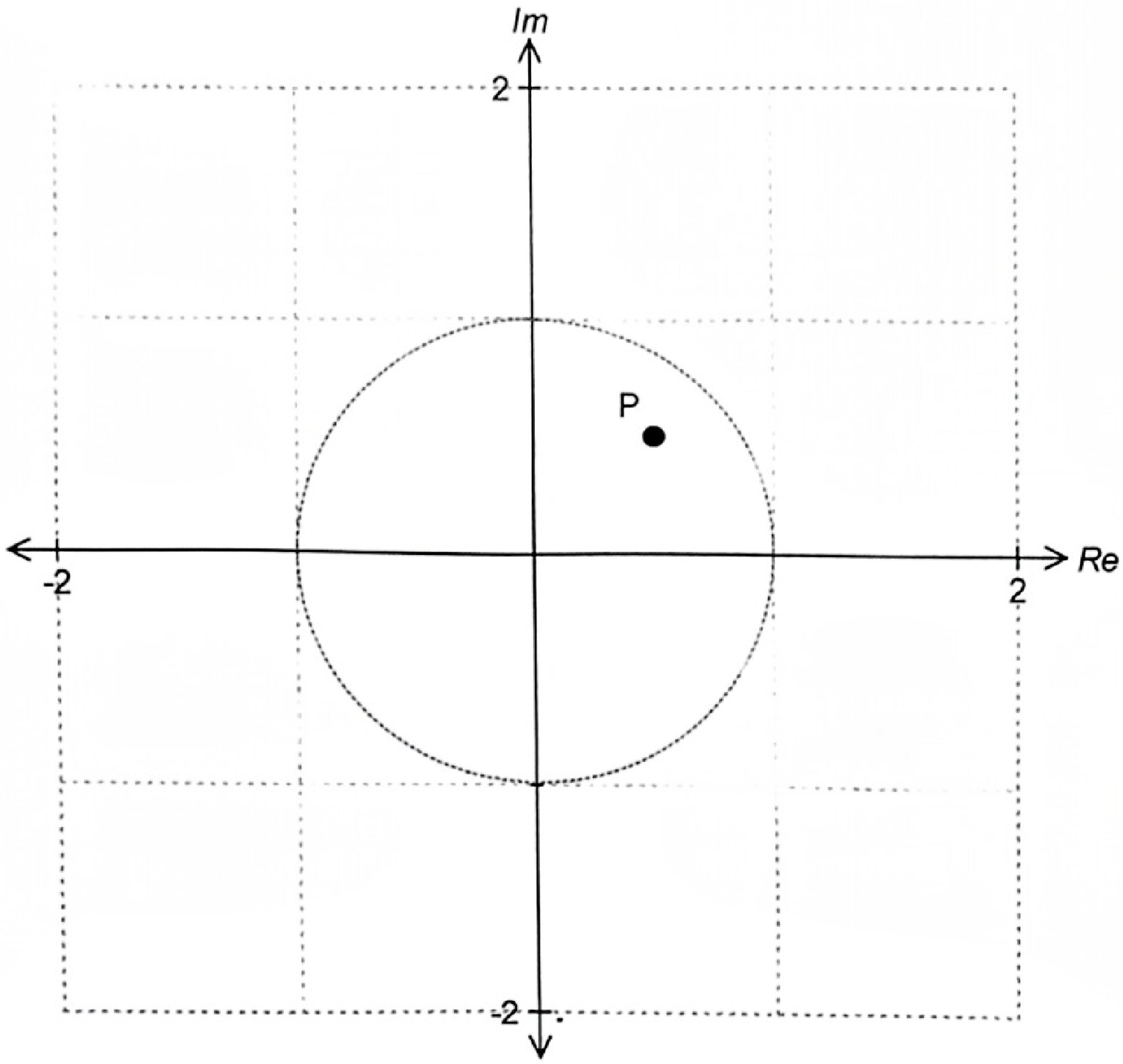


Describe the complex numbers represented by each of the points Q, R, S and T using the complex numbers u and v and/or their conjugates.
For example, the point P represents $v - u$.

Calculator Free

7. [11 marks: 7, 2, 2] [TISC]

(a) The complex number z where $|z| < 1$, is represented by the point P as marked in the Argand diagram below.

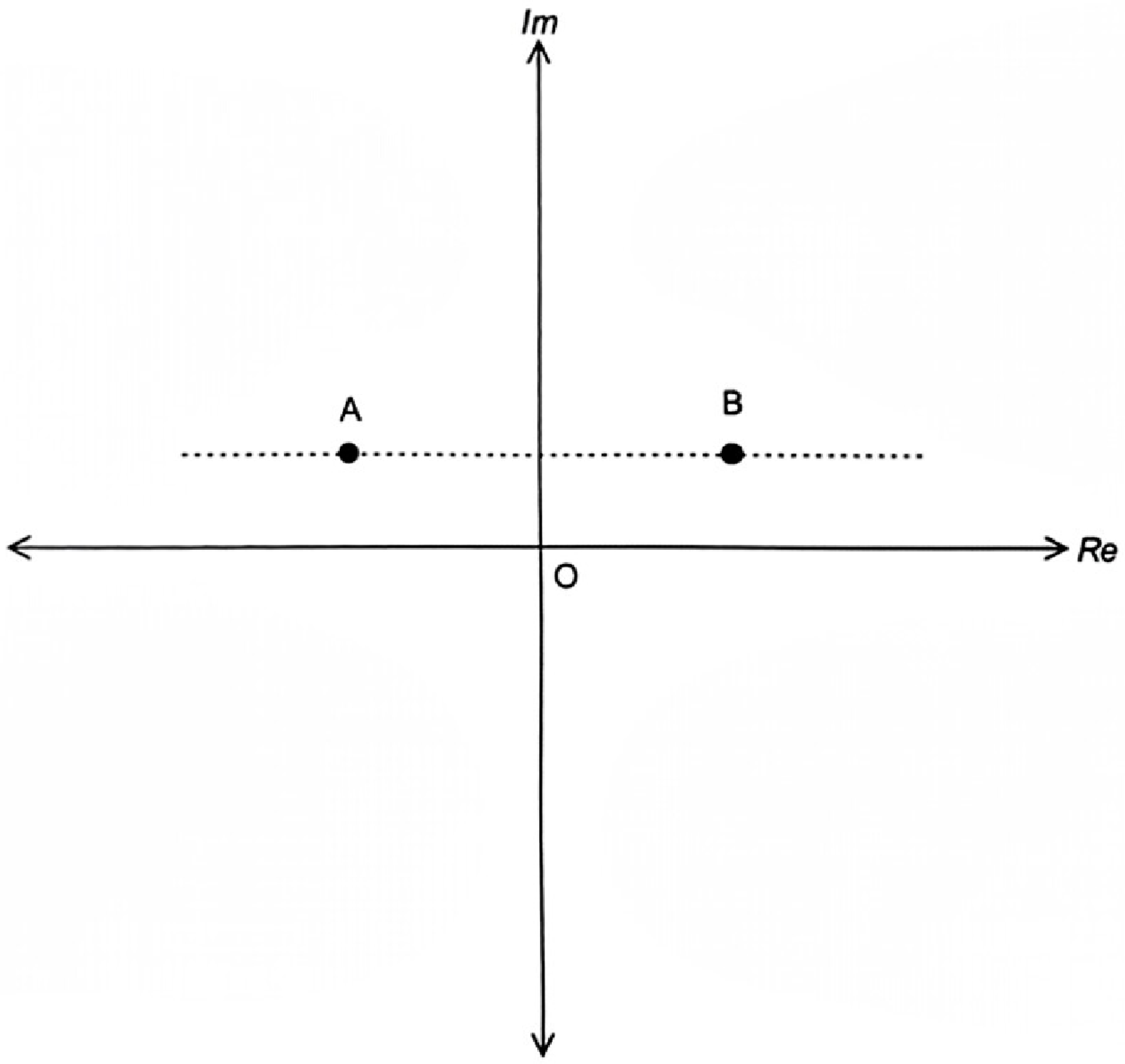


Mark clearly on the diagram above the points representing the complex numbers:

(i) $\bar{z}$ (ii) z^2 (iii) $\sqrt{z}$ (iv) $\dfrac{1}{z}$.

Calculator Free

7. (b) The complex numbers z_1 and z_2 are represented by the points A and B in the Argand diagram below. The complex numbers z_1 and z_2 can also be represented by the vectors **OA** and **OB** respectively.



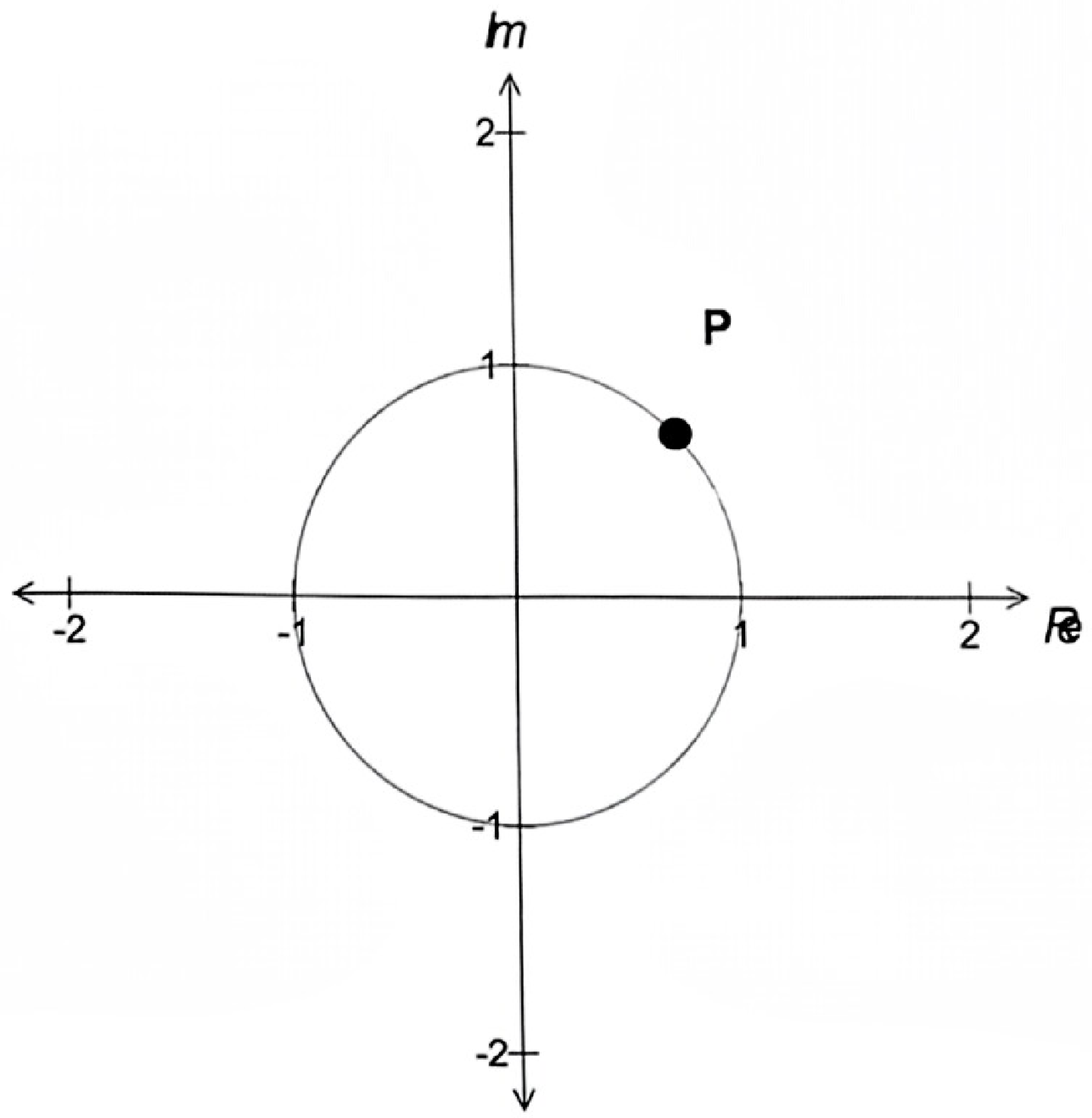
(i) Describe the vector **AB** using the complex numbers z_1 and z_2 where appropriate.

(ii) z is a complex number represented by the point Z such that $z_1 - z_2 = iz$. Mark on the Argand diagram above the position(s) of the point Z.

Calculator Free

8. [13 marks: 8, 5] [TISC]

(a) The complex number z where $|z| = 1$, is represented by the point P as marked in the Argand diagram below.

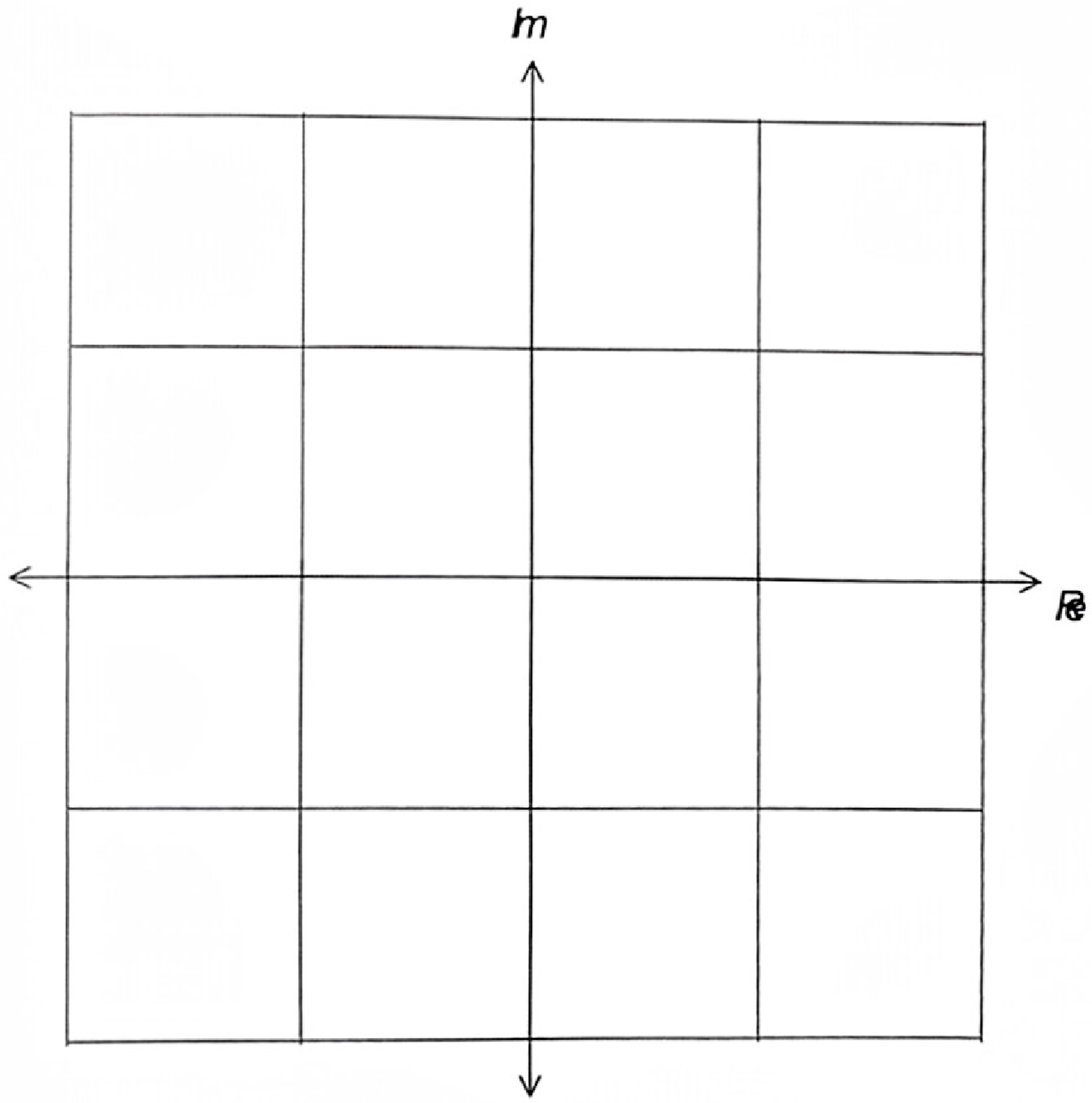


Mark clearly on the diagram above the points representing the complex numbers:

(i) $-z$ (ii) $-i\,z$ (iii) $z + \overline{z}$ (iv) $z \times \overline{z}$

Calculator Free

8. (b) The locus of the complex number z satisfies the equation $|z-1| = |\bar{z}|$.
Find the Cartesian equation of the locus and hence sketch the locus of z in the Argand diagram provided below.



Calculator Free

9. [7 marks: 3, 2, 2] [TISC]

The complex number z is defined by $z = \dfrac{a+4i}{i} + \dfrac{4}{1+i}$ where a is a real constant.

(a) Rewrite z in the form $x + y\,i$ where x and y are real.

(b) Find the value of a if z lies on the line $Im(z) = -Re(z)$.

(c) Show that z cannot lie on the curve $arg(z) = \dfrac{3\pi}{4}$.

Calculator Assumed

10. [8 marks: 3, 5] [TISC]

Let $w = x + y\,i$.

(a) If $\left|\dfrac{w}{1-w}\right| = 1$, show that w lies on the line with equation $x = \dfrac{1}{2}$.

(b) If $\left|\dfrac{w}{1-w}\right| = 3$, show that w lies on a circle. Find the equation of this circle.

Calculator Assumed

11. [8 marks]

The locus of the complex number z satisfies the equation $\left|\dfrac{z-1+2i}{z-1-2i}\right| = 2$.

Find the Cartesian equation of the locus. Hence sketch the locus of z.

Calculator Assumed

13. [6 marks: 2, 1, 1, 2]

Let $a = 1 + i$ and $b = 1 + i\sqrt{3}$.

(a) Express a and b in exact *cis* form.

$a = 1 + i = \sqrt{2}\ cis\ \frac{\pi}{4}$ ✓

$b = 1 + i\sqrt{3} = 2\ cis\ \frac{\pi}{3}$ ✓

(b) Find $\frac{b}{a}$ in exact *cis* form.

$$\frac{b}{a} = \frac{2\ cis\left(\frac{\pi}{3}\right)}{\sqrt{2}\ cis\left(\frac{\pi}{4}\right)} = \sqrt{2}\ cis\left(\frac{\pi}{12}\right)$$ ✓

(c) Find $\frac{b}{a}$ in exact Cartesian form.

$$\frac{b}{a} = \frac{1+i\sqrt{3}}{1+i} = \left(\frac{\sqrt{3}+1}{2}\right)+\left(\frac{\sqrt{3}-1}{2}\right)i$$ ✓

(d) Use your answers in (b) and (c) to find $\cos\frac{\pi}{12}$ in exact form.

$$\sqrt{2}\ cis\left(\frac{\pi}{12}\right) = \left(\frac{\sqrt{3}+1}{2}\right)+\left(\frac{\sqrt{3}-1}{2}\right)i$$

$$cis\left(\frac{\pi}{12}\right) = \frac{1}{\sqrt{2}}\left[\left(\frac{\sqrt{3}+1}{2}\right)+\left(\frac{\sqrt{3}-1}{2}\right)i\right]$$ ✓

$$\cos\frac{\pi}{12} + i\sin\frac{\pi}{12} = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)+i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

Compare real parts:

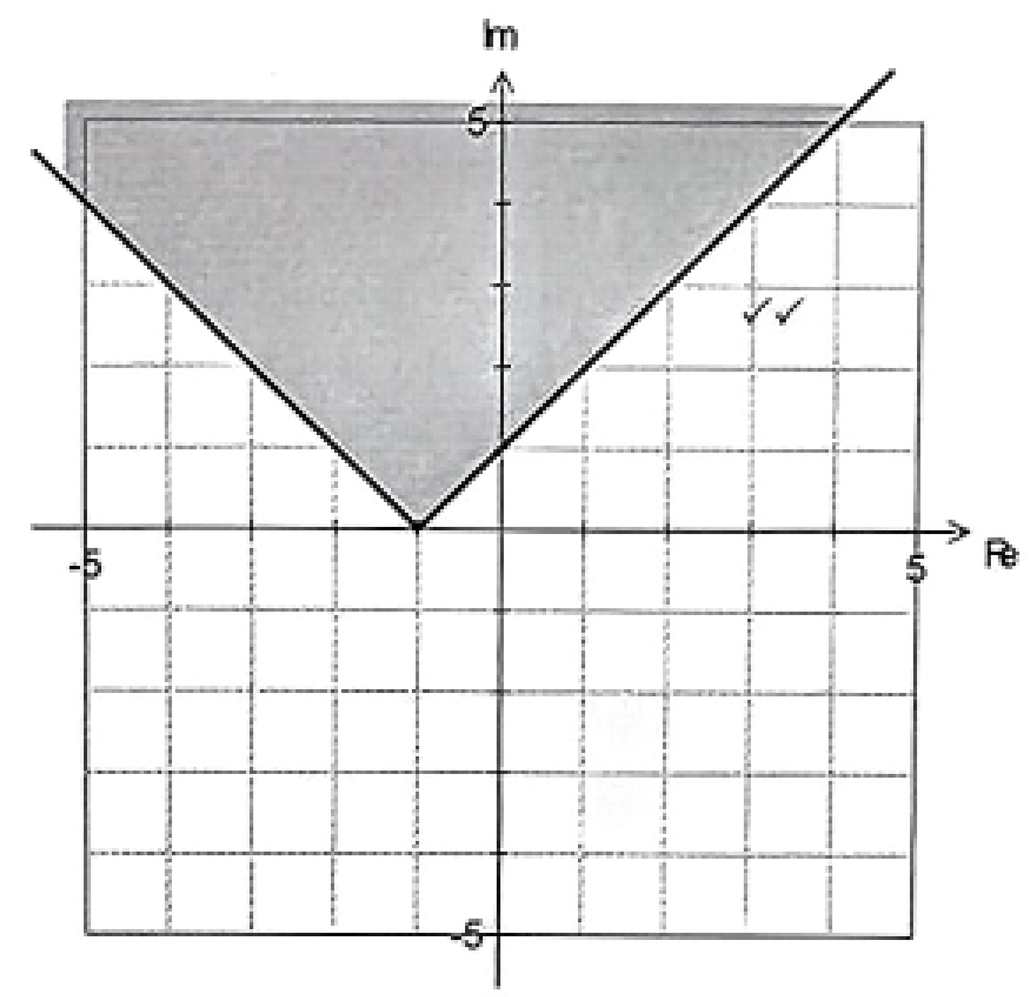
$$\cos\frac{\pi}{12} = \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$ ✓

02 Complex Numbers II

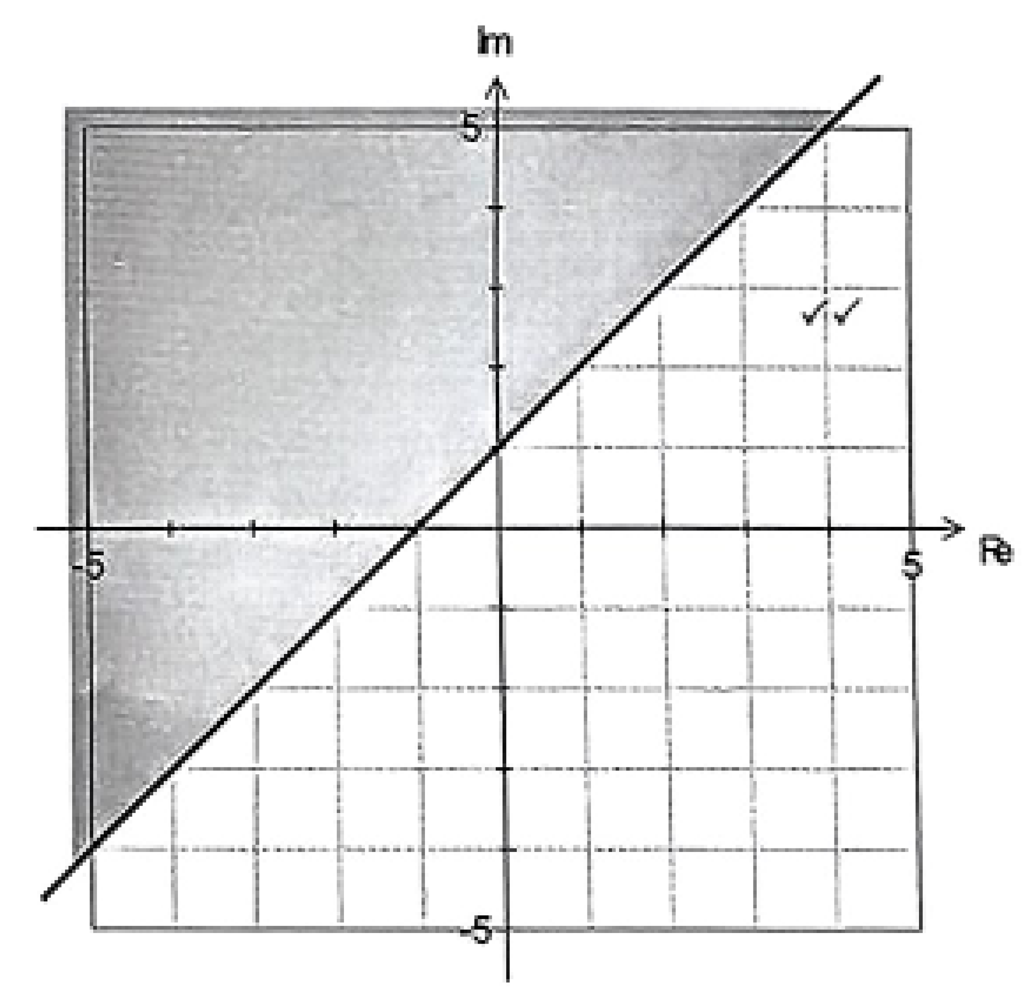
Calculator Free

1. [13 marks: 2, 2, 2, 3, 4] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{z : \text{Im}(z) \ge |\text{Re}(z) + 1|\}$.



(b) Sketch the region in the Argand Plane defined by $\{ z : |z-1| \ge |z+1-2i| \}$

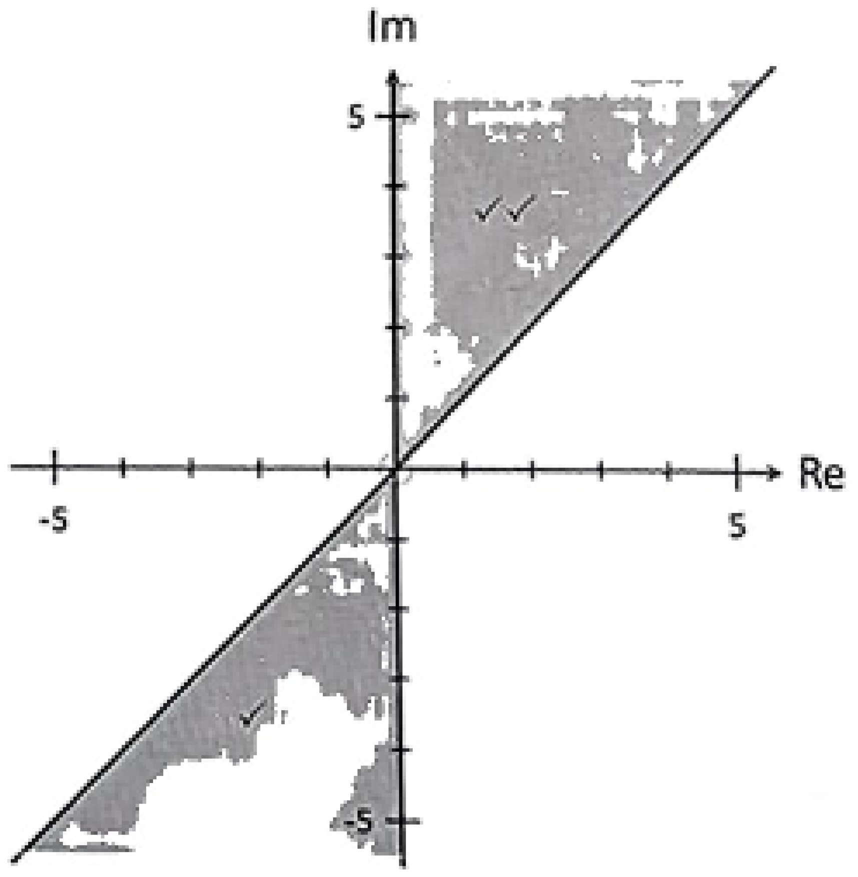


Calculator Free

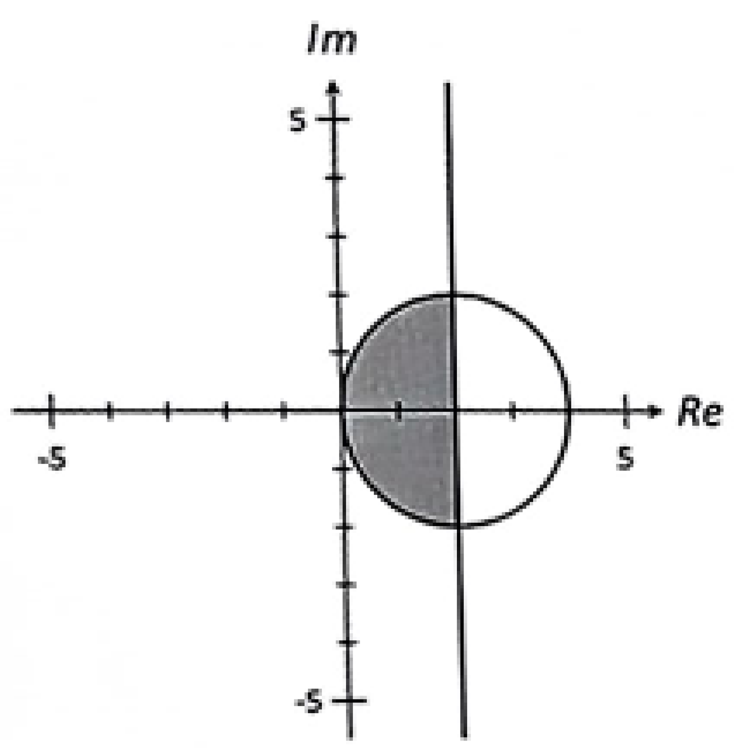
1. (c) (i) Let $z = r \, cis \, \theta$ where $0 < \theta \le \pi$. Show that $\text{Arg}(z^2) = 2\theta$.

$z^2 = r^2 \, cis \, 2\theta$ ✓
$\Rightarrow \; arg(z^2) = 2\theta$ ✓

(ii) Sketch the region in the Argand Plane defined by $\{z : \text{Arg}(z^2) \ge \frac{\pi}{2}\}$



(d) Describe the complex set that defines the region in the Argand Plane shown below.

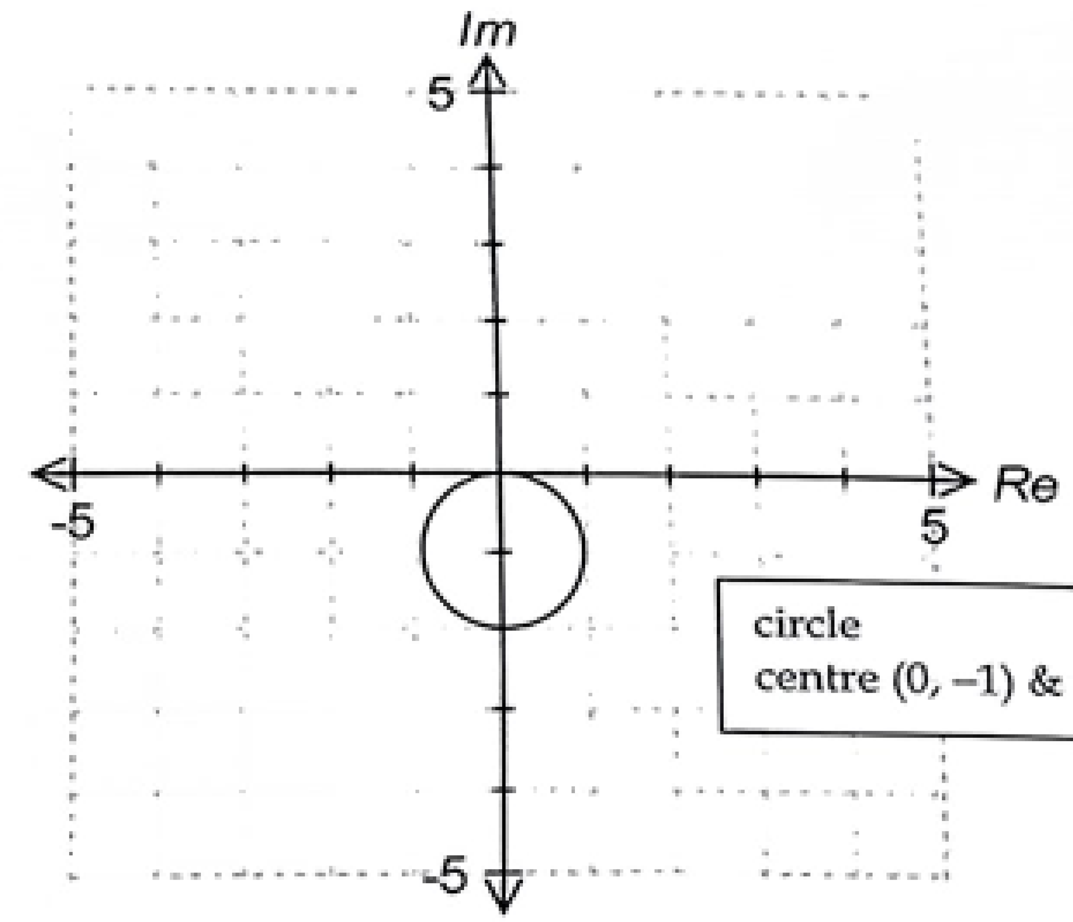


$\{z : |z - 2| \le 2 \text{ and } \text{Re}(z) \le 2\}$
✓ ✓ ✓ ✓

Calculator Free

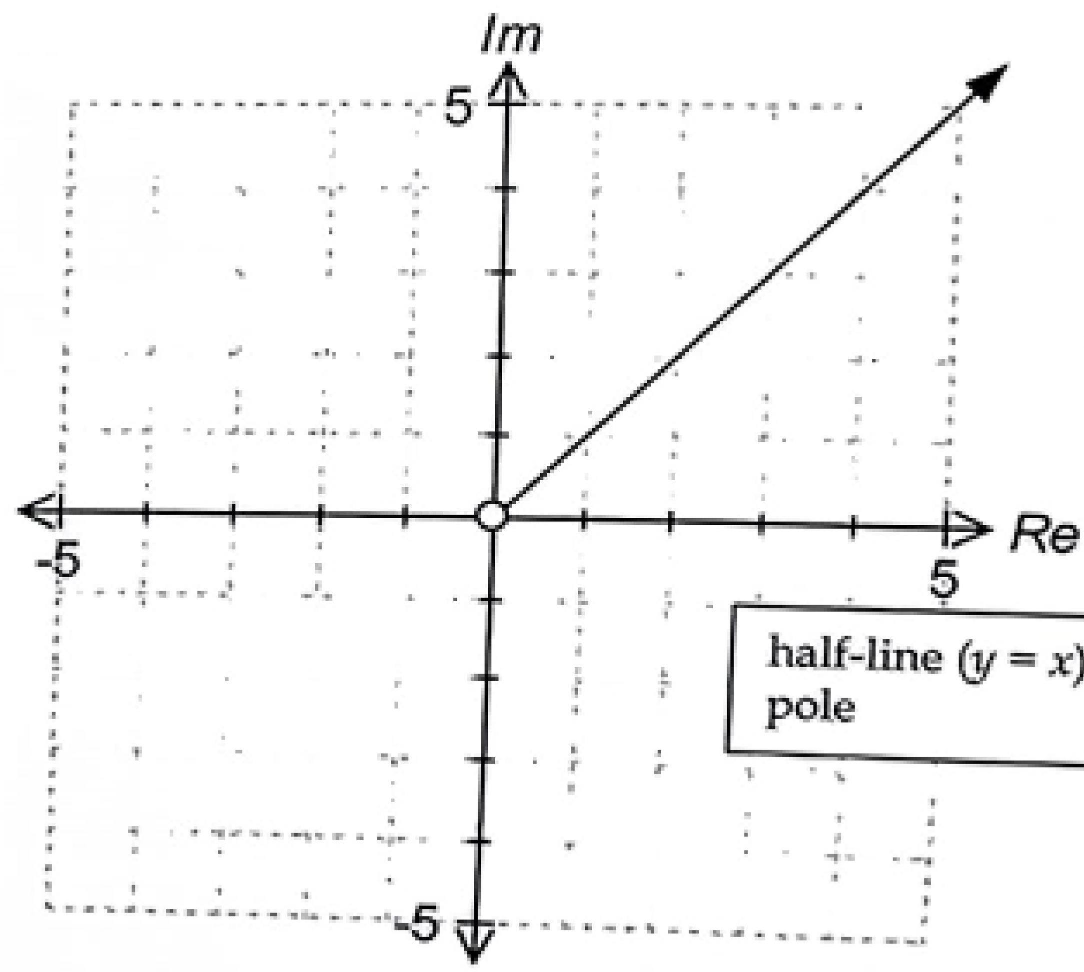
2. [8 marks: 2, 2, 2, 2] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{z : |z + i| = 1\}$.



circle ✓
centre (0, −1) & radius 1 ✓

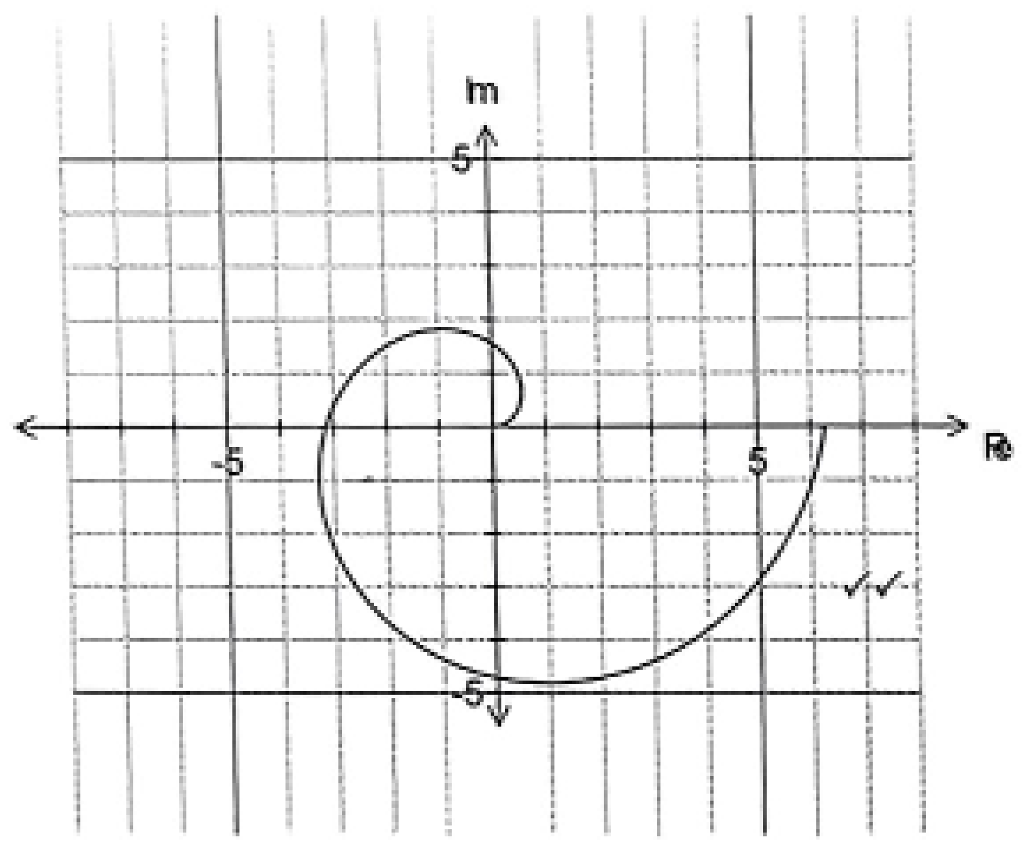
(b) Sketch the region in the Argand Plane defined by $\{ z : arg\,(z) = \frac{\pi}{4} \}$.



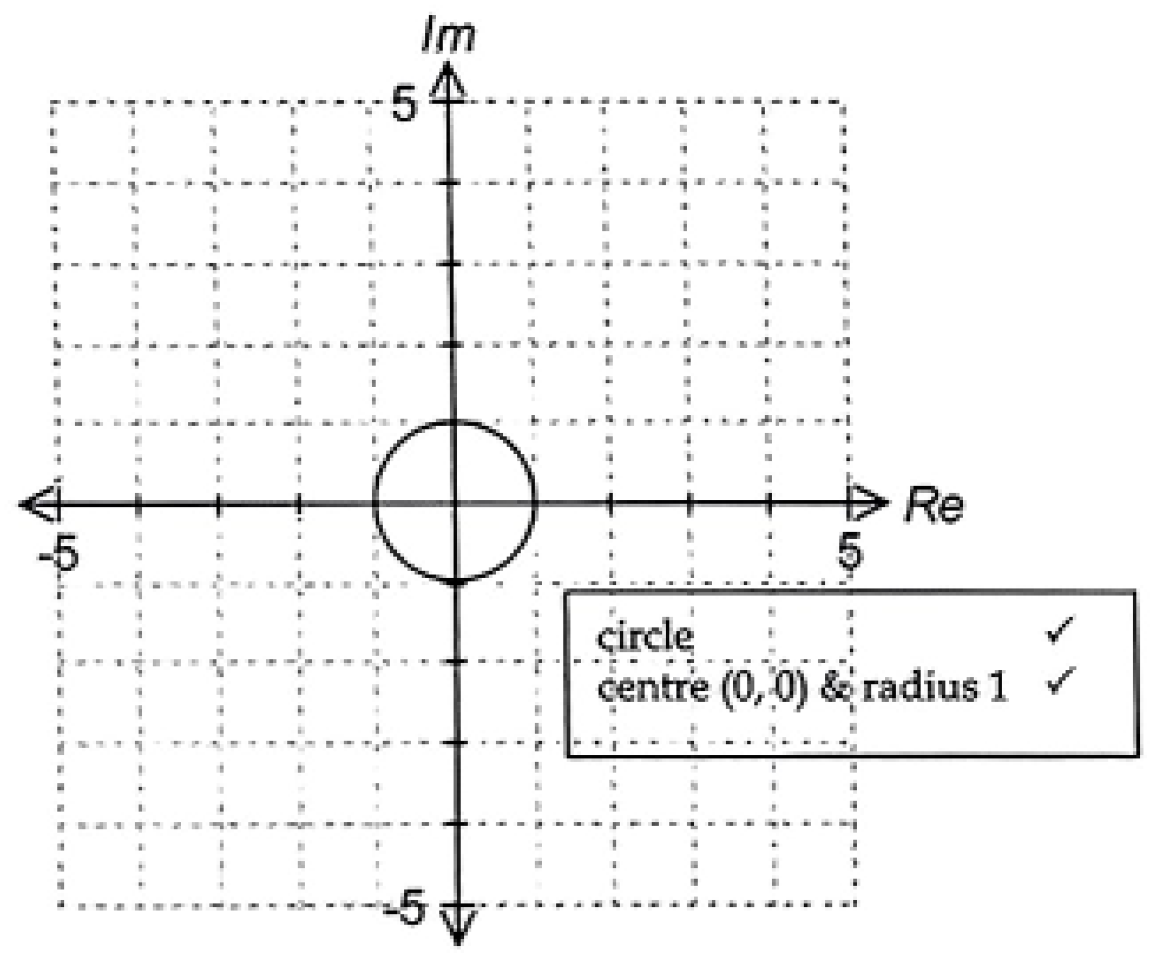
half-line ($y = x$) ✓
pole ✓

Calculator Free

2. (c) Sketch the region in the Argand Plane defined by
$\{ z : | z | = arg\ (z)$ where $0 \le arg\ (z) \le 2\pi \}$.



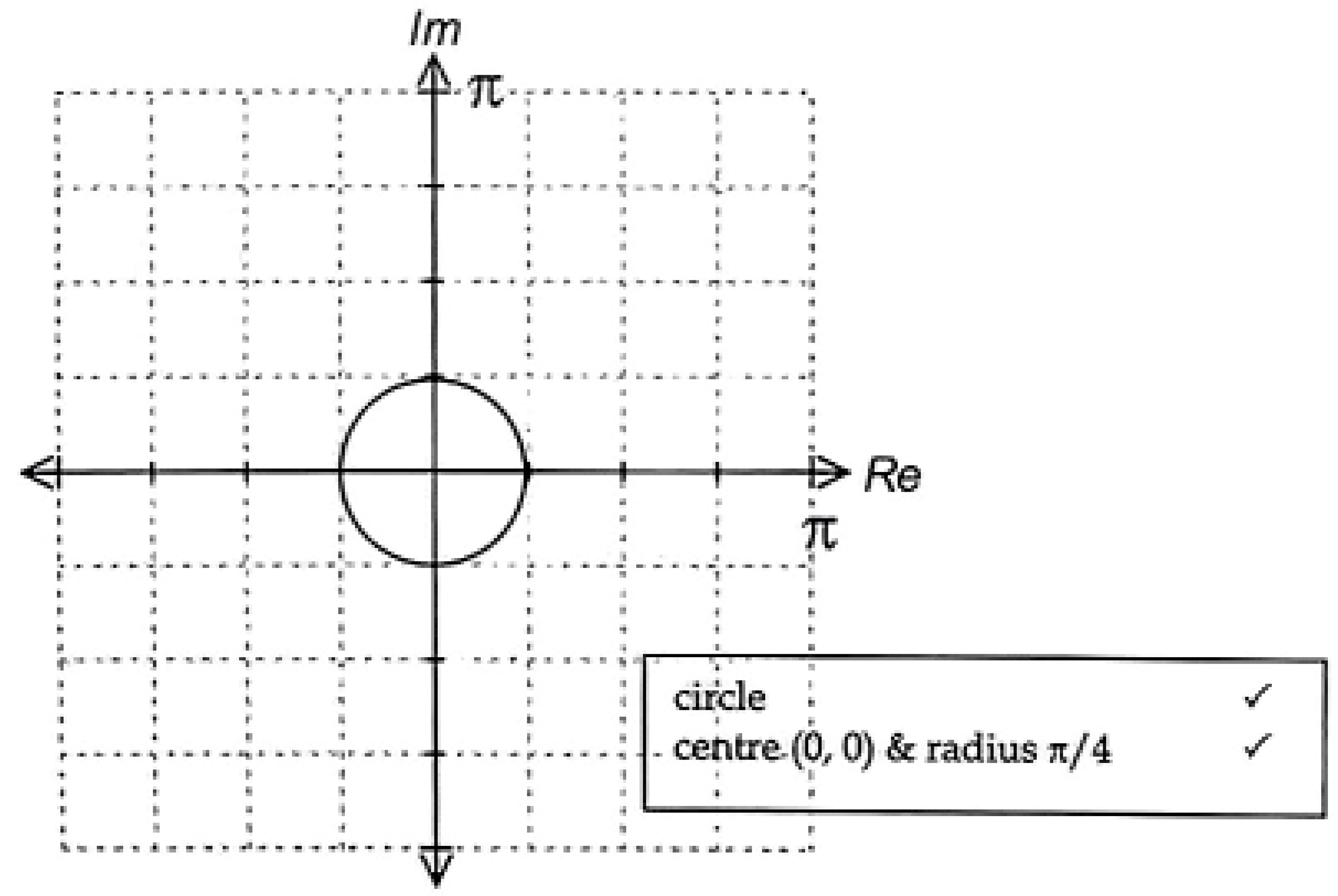
(d) Sketch the region in the Argand Plane defined by
$\{z : z = \cos\theta + i \sin\theta$ where $-\pi < \theta \le \pi\}$.



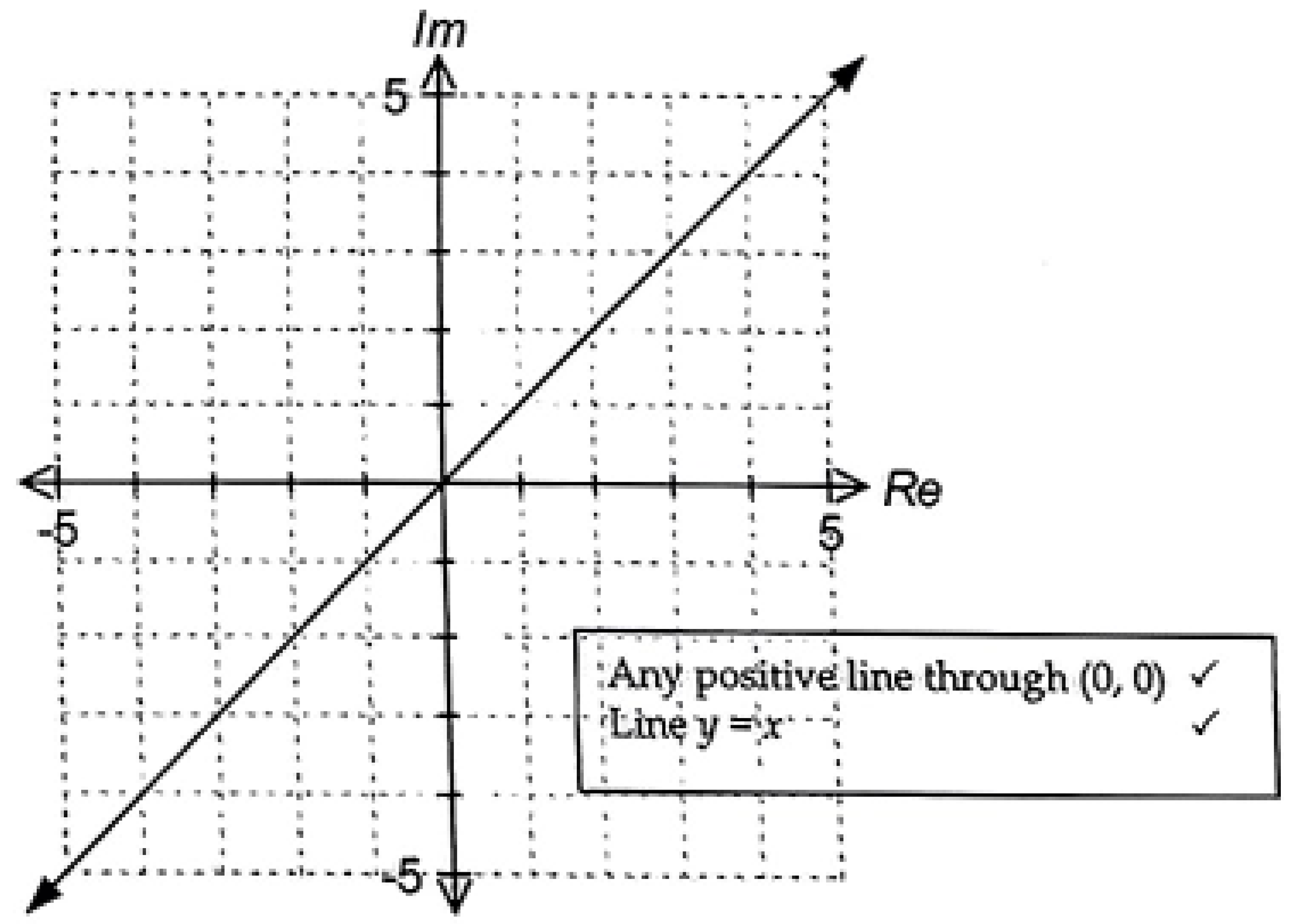
Calculator Free

3. [10 marks: 2, 2, 3, 3] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{ z : | z | = \frac{\pi}{4} \}$.



(b) Sketch the region in the Argand Plane defined by $\{z : \tan [arg(z)] = 1\}$.

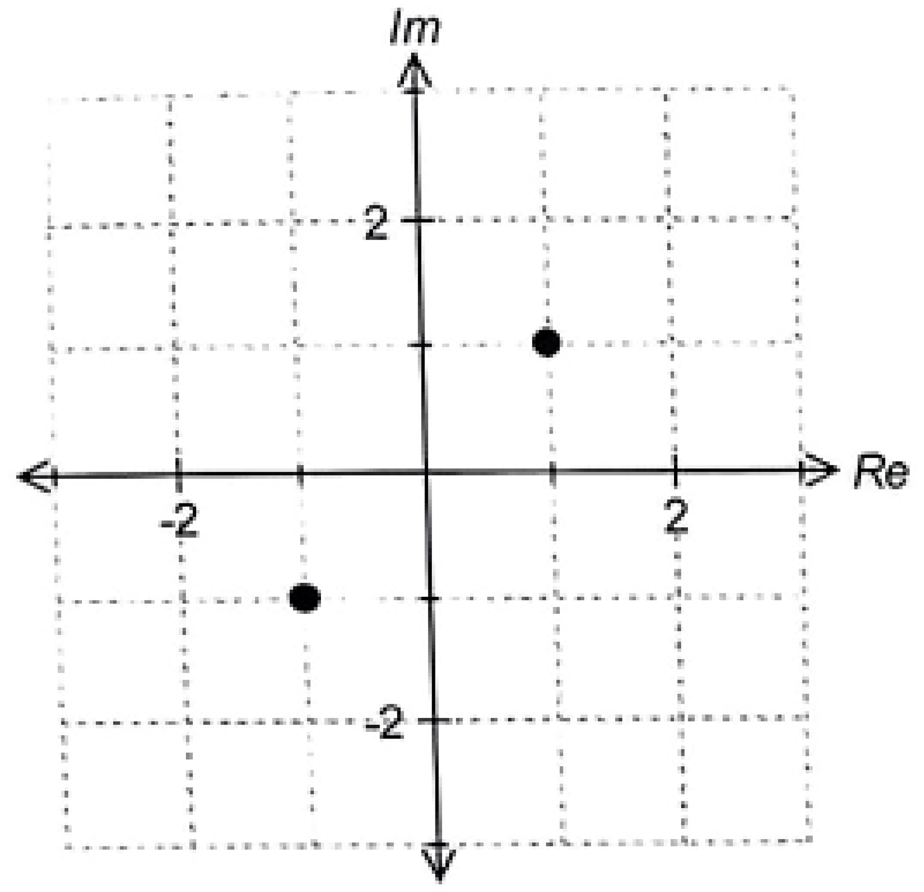


Calculator Free

3. (c) Consider the region in the Argand Plane defined by $\{z : z^2 = 2i\}$.
   Let $z = x + iy$ where x and y are real numbers.
   (i) Show that the Cartesian equation of this region is given by $x^4 = 1$.

$(x + iy)^2 = 2i$
$(x^2 - y^2) + 2xyi = 2i$
$\Rightarrow x^2 - y^2 = 0$ and $xy = 1$ ✓✓
$x^2 - \left(\frac{1}{x}\right)^2 = 0$
$\Rightarrow x^4 = 1$ ✓

   (ii) Hence, show that this region consists of exactly *two* points.
   Mark these two points clearly on the axes below.

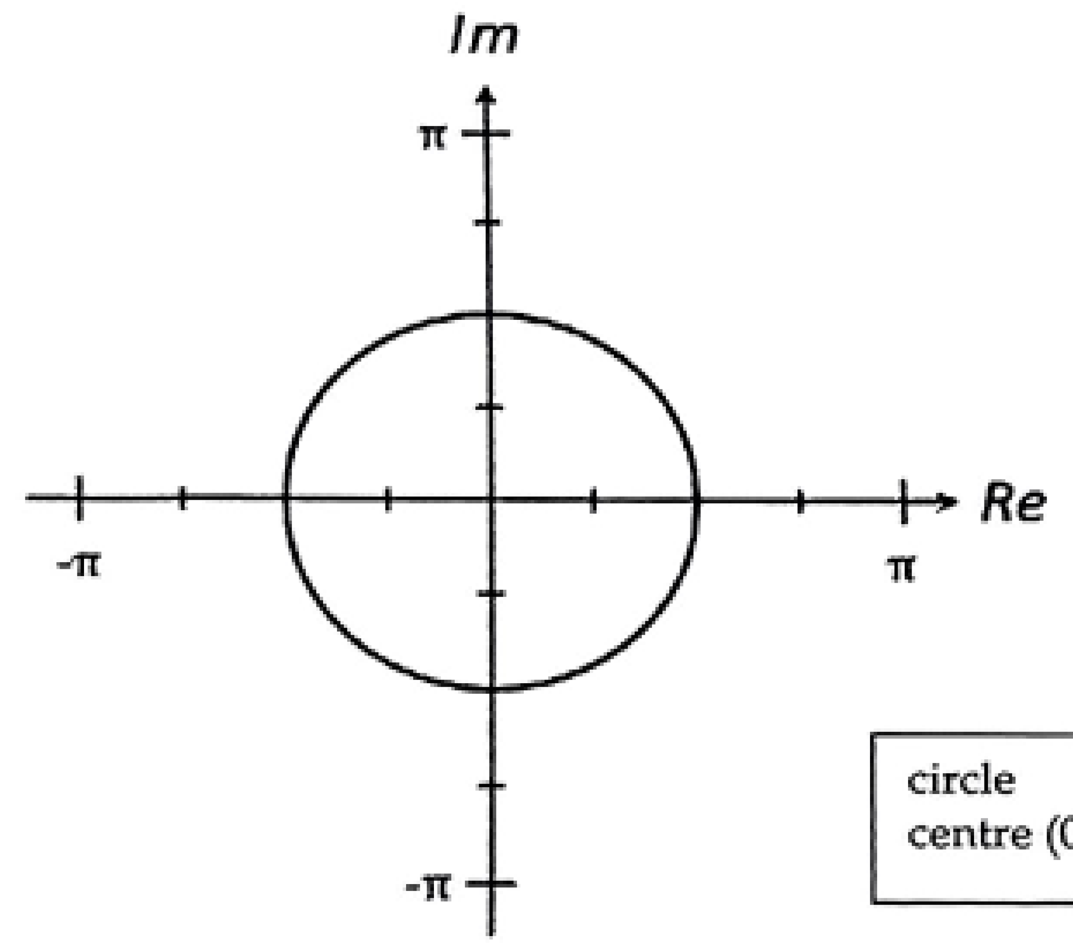


Since, x is real, $x^4 = 1 \Rightarrow x = \pm 1$. ✓
Hence, points are (1, 1) and (−1, −1). ✓✓

Calculator Free

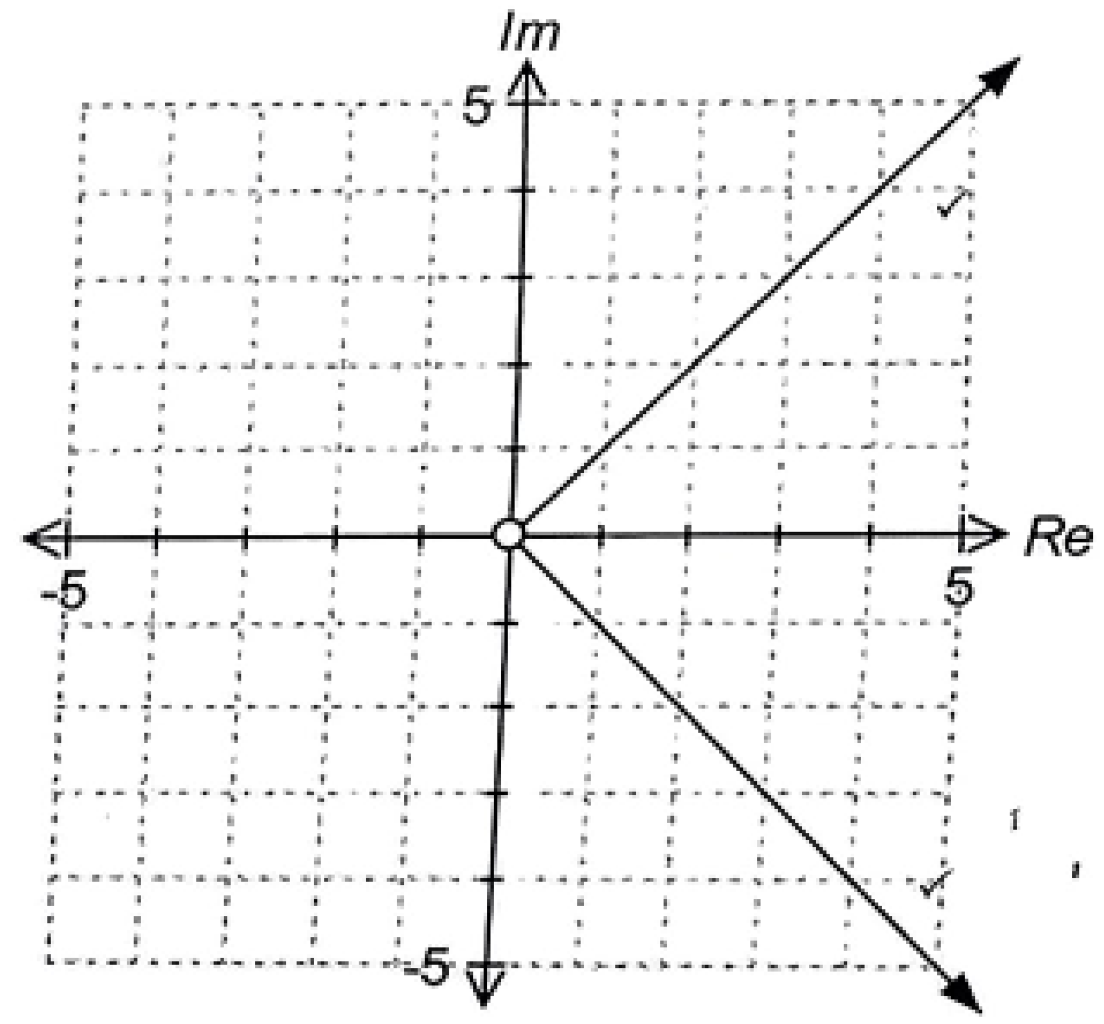
4. [10 marks: 2, 2, 3, 3] [TISC]

(a) Sketch the region in the Argand Plane defined by $\{ z : |z| = \frac{\pi}{2} \}$.



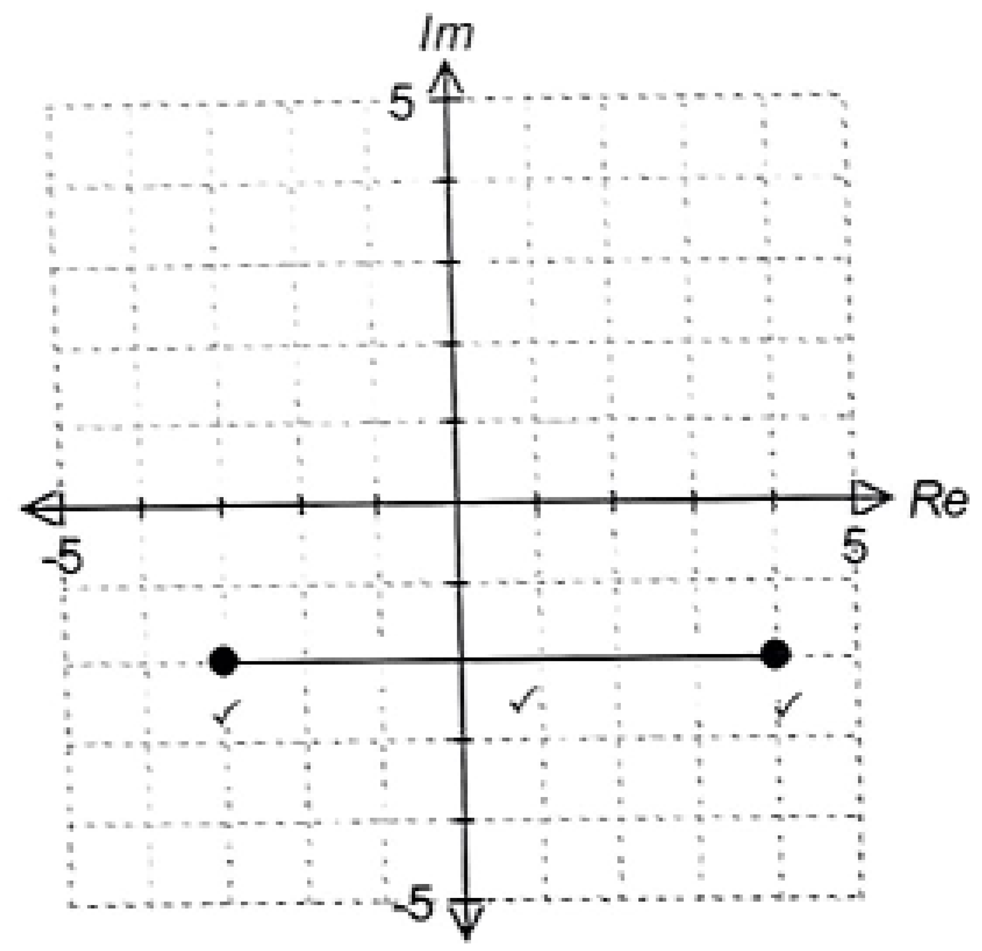
circle ✓
centre (0, 0) & radius π/2 ✓

(b) Sketch the region in the Argand Plane defined by $\{ z : |arg(z)| = \frac{\pi}{4} \}$.

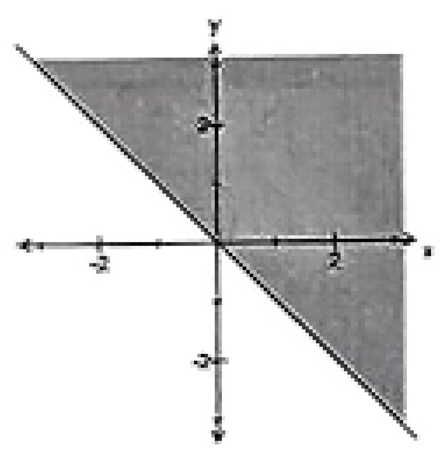


Calculator Free

4. (c) Sketch on the diagram below the locus of the point z defined by:
$\{z : |z + 3 + 2i| + |z - 4 + 2i| = 7\}$.



(d) Region R in the Argand Plane is defined by $\{ z : |z - 1| \le |z + i| \}$.
Region R can also be described in Cartesian form by the inequality $ax + by \ge 0$. Find a and b. [Hint: Use a sketch.]



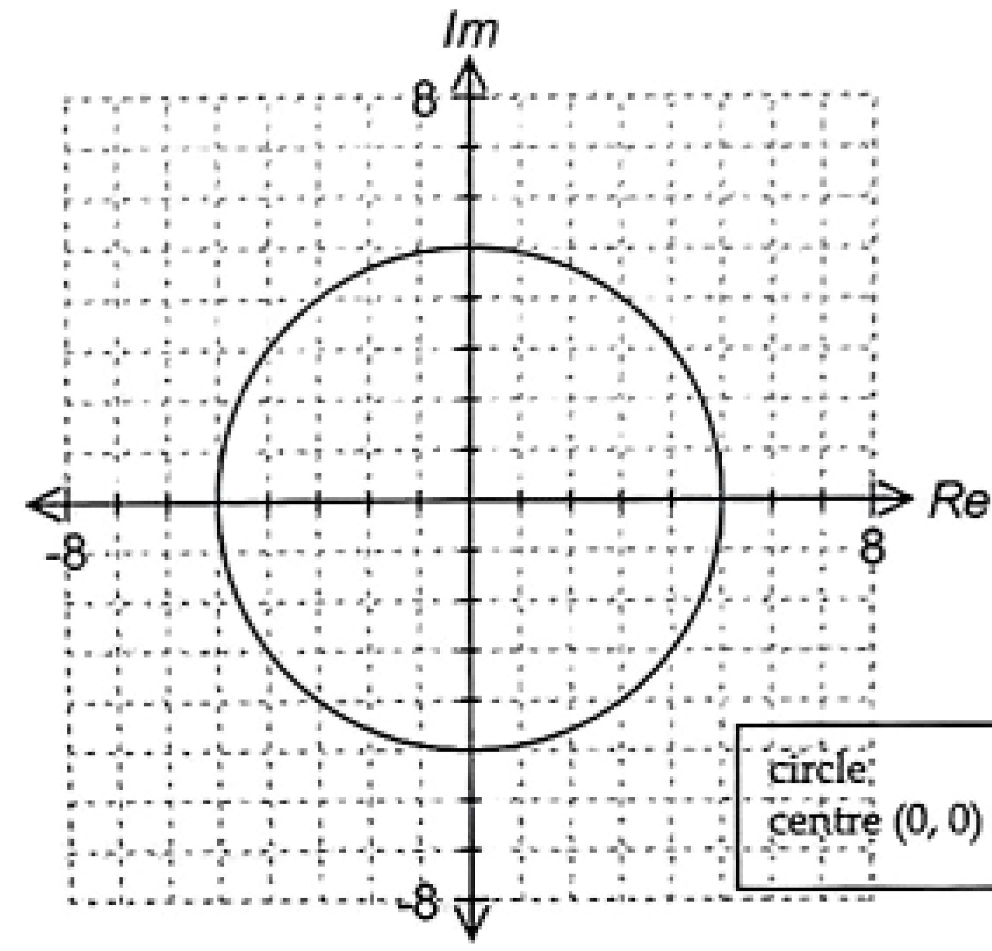
Shaded region is $x + y \ge 0$ ✓ ✓

Hence, $a = 1, b = 1$ ✓

Calculator Free

5. [11 marks: 2, 2, 3, 4] [TISC]

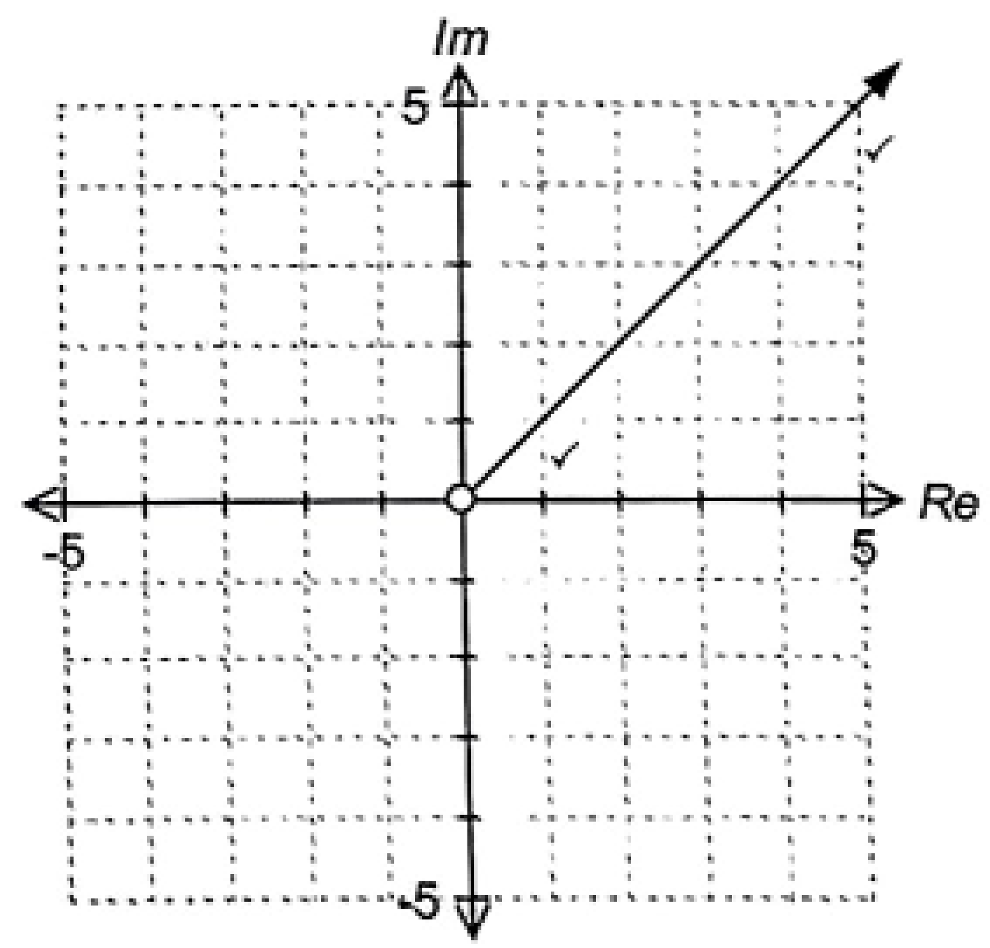
(a) Sketch the region in the Argand Plane defined by $\{ z : |\bar{z}| = 5\}$.



circle ✓

centre (0, 0) & radius 5 ✓

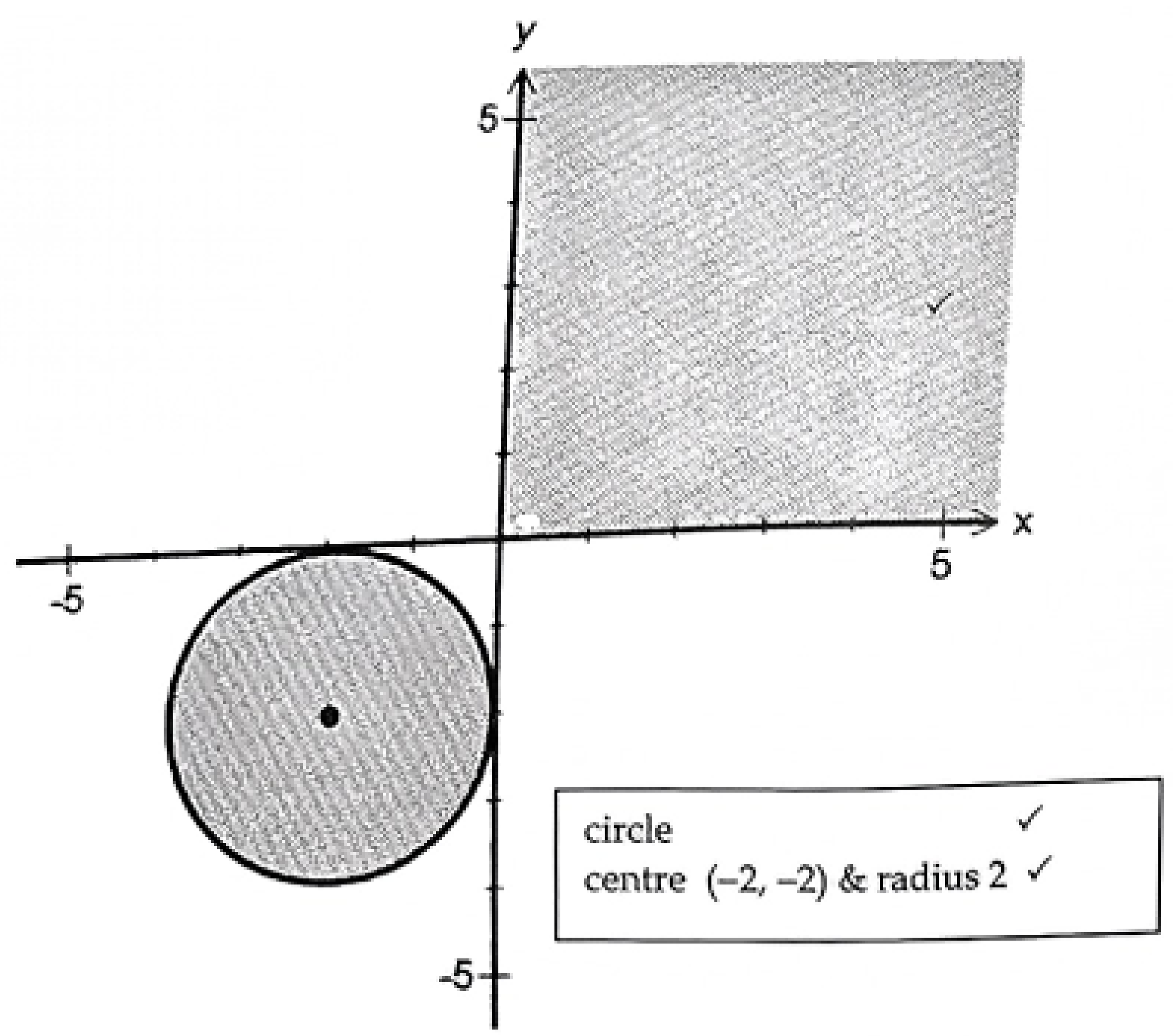
(b) Sketch the region in the Argand Plane defined by $\{ z : arg(\bar{z}) = \frac{-\pi}{4} \}$.



Calculator Free

5. (c) Sketch on the diagram below the locus of the point z defined by:

$\{z : |z + 2 + 2i| \le 2 \cup 0 \le arg\,(z) \le \frac{\pi}{2}\}$.



(d) Find, in its simplest form the Cartesian equation of the locus of the point z defined by $|z - 1 - i| = \mathbf{Re}\,(z + 3 + 4i)$.

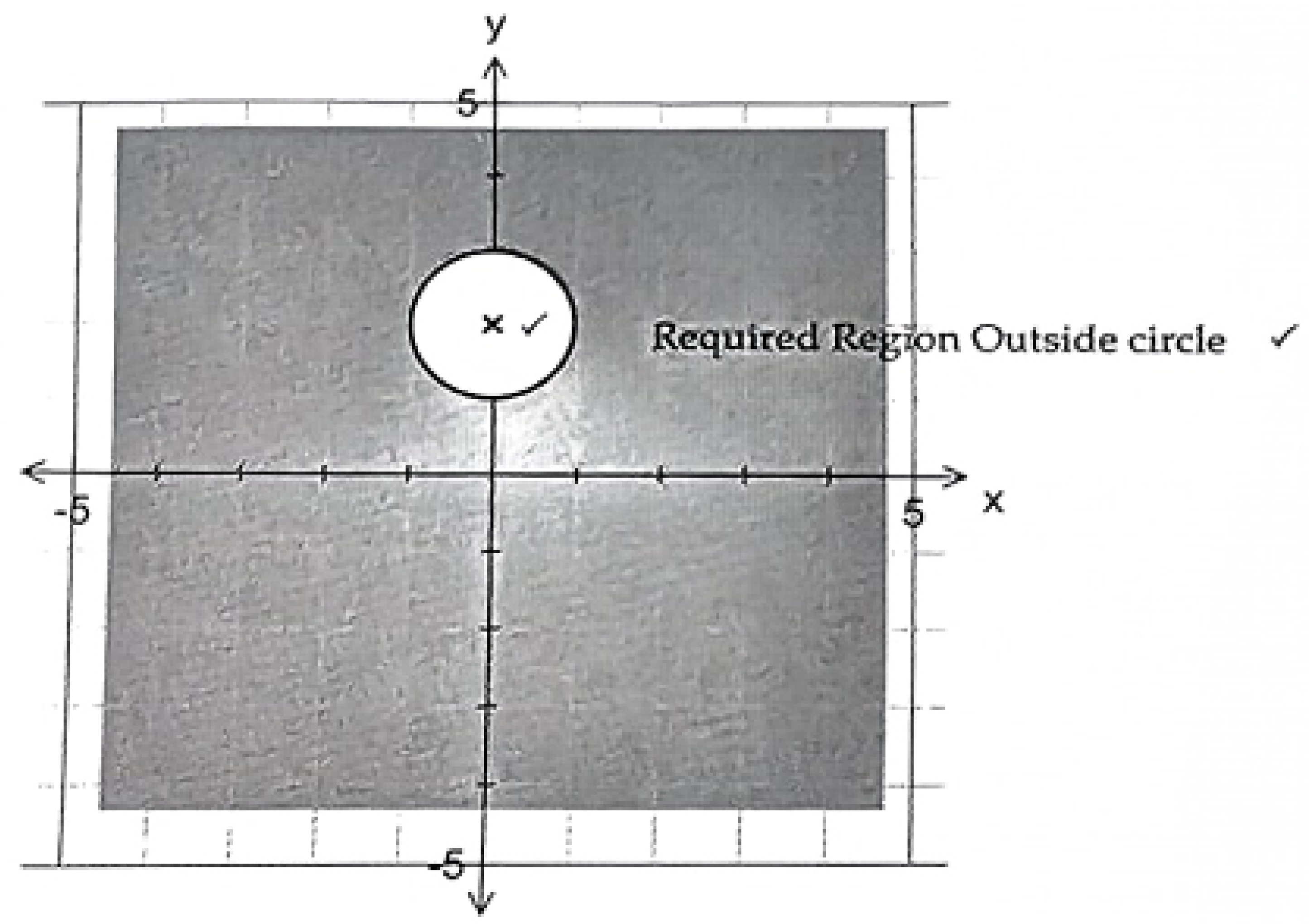
$\lvert (x-1) + (y-1)i \rvert = \text{Re}(x + 3 + 4i)$	✓
$\sqrt{[(x-1)^2 + (y-1)^2]} = x + 3$	✓
$(x-1)^2 + (y-1)^2 = (x+3)^2$	✓
$y^2 - 2y - 8x - 7 = 0$	✓

Calculator Free

6. [12 marks: 2, 3, 7] [TISC]

(a) Sketch on the diagram below the locus of the point z defined by:

$\{z : |z - 2i| \ge 1\}$.

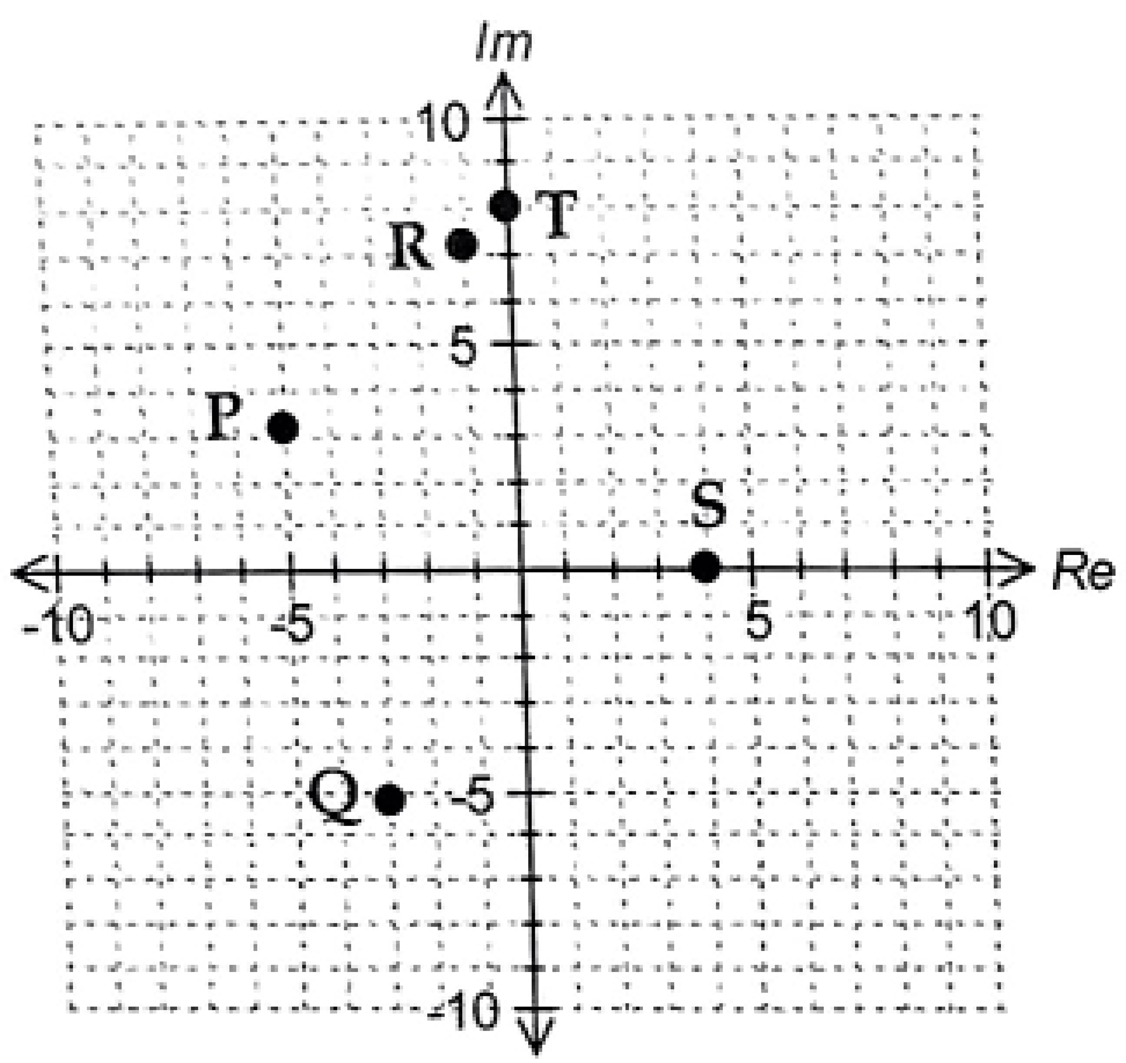


(b) Find, in simplest form the Cartesian equation of the locus of the point z defined by $|z - 1| = |z - 1 + 2i|$.

Let $z = x + yi$.	
$\lvert (x-1) + yi \rvert = \lvert (x-1) + (y+2)i \rvert$	✓
$(x-1)^2 + y^2 = (x-1)^2 + (y+2)^2$	✓
$y^2 = y^2 + 4y + 4$	
$y = -1$	✓

Calculator Free

6. (c) Consider the complex numbers $u = 2 + 2i$ and $v = -3 + 3\sqrt{3}\,i$.
The Argand diagram below shows the points P, Q, R, S and T.



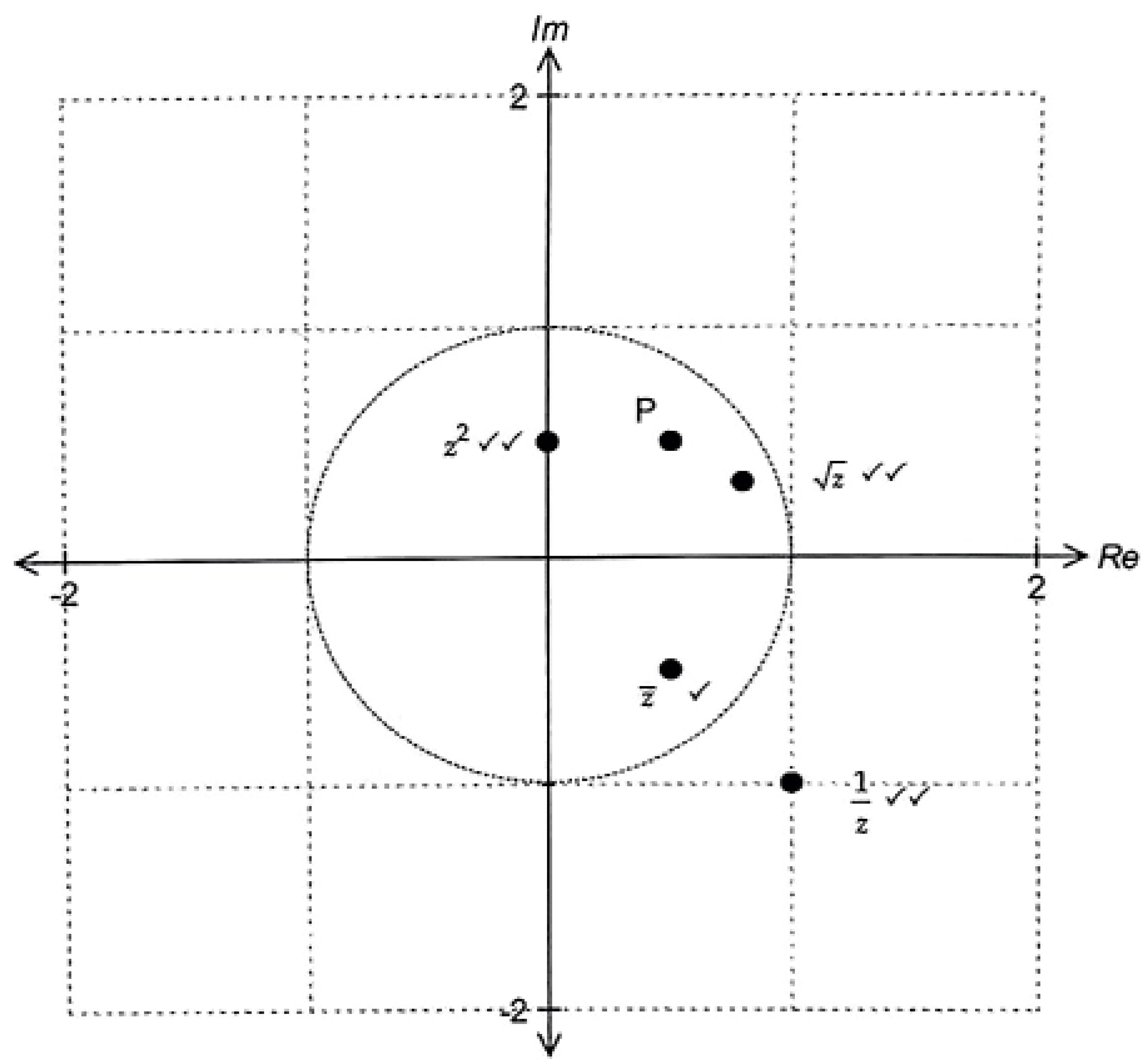
Describe the complex numbers represented by each of the points Q, R, S and T using the complex numbers u and v and/or their conjugates.
For example, the point P represents $v - u$.

Q: $\bar{v}$	✓
R: $u + v$	✓✓
S: $u + \bar{u}$	✓✓
T: u^2 or $2(u - \bar{u})$	✓✓

Calculator Free

7. [11 marks: 7, 2, 2] [TISC]

(a) The complex number z where $|z| < 1$, is represented by the point P as marked in the Argand diagram below.

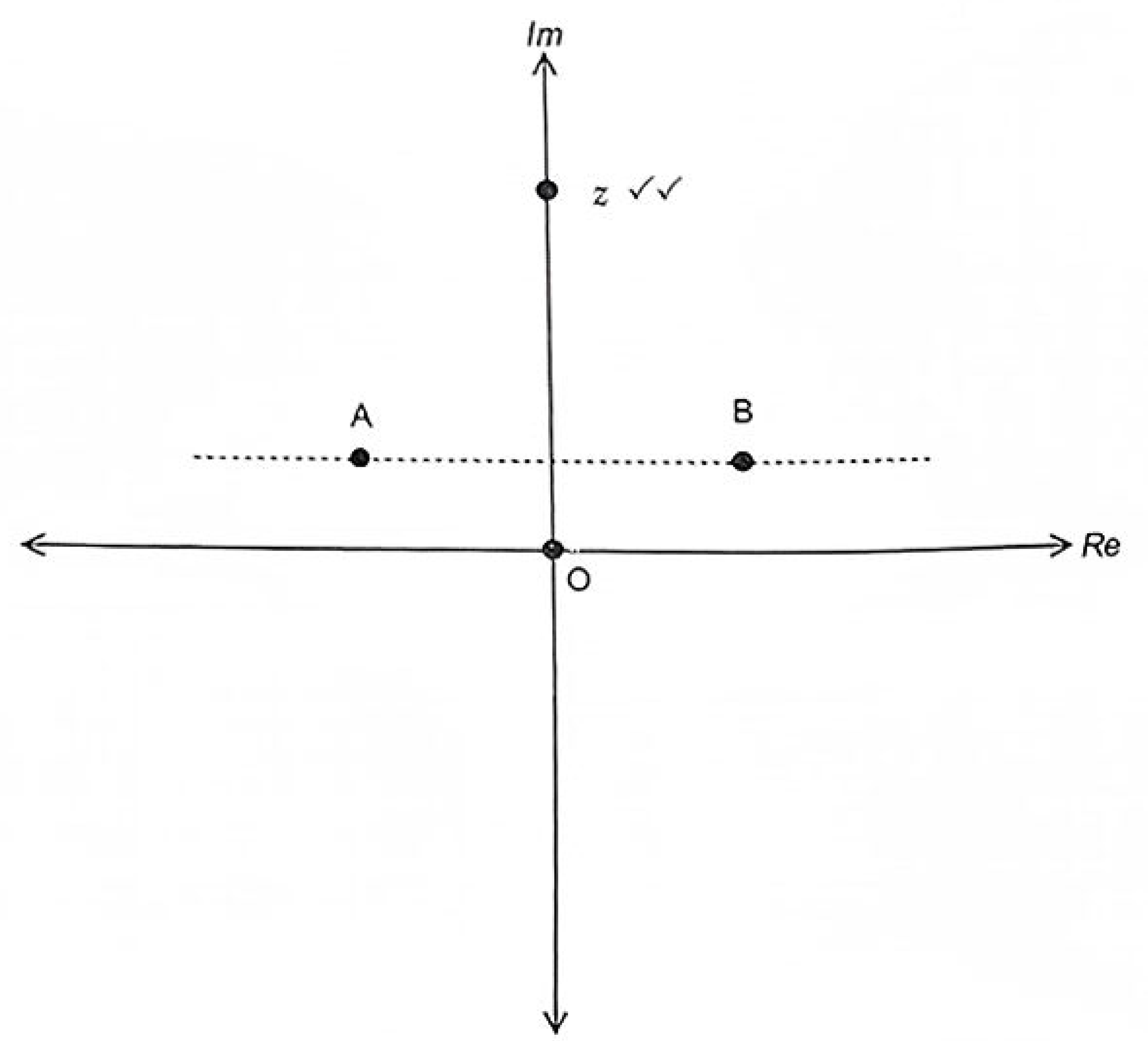


Mark clearly on the diagram above the points representing the complex numbers:

(i) $\bar{z}$ (ii) z^2 (iii) $\sqrt{z}$ (iv) $\frac{1}{z}$.

Calculator Free

7. (b) The complex numbers z_1 and z_2 are represented by the points A and B in the Argand diagram below. The complex numbers z_1 and z_2 can also be represented by the vectors **OA** and **OB** respectively.



(i) Describe the vector **AB** using the complex numbers z_1 and z_2 where appropriate.

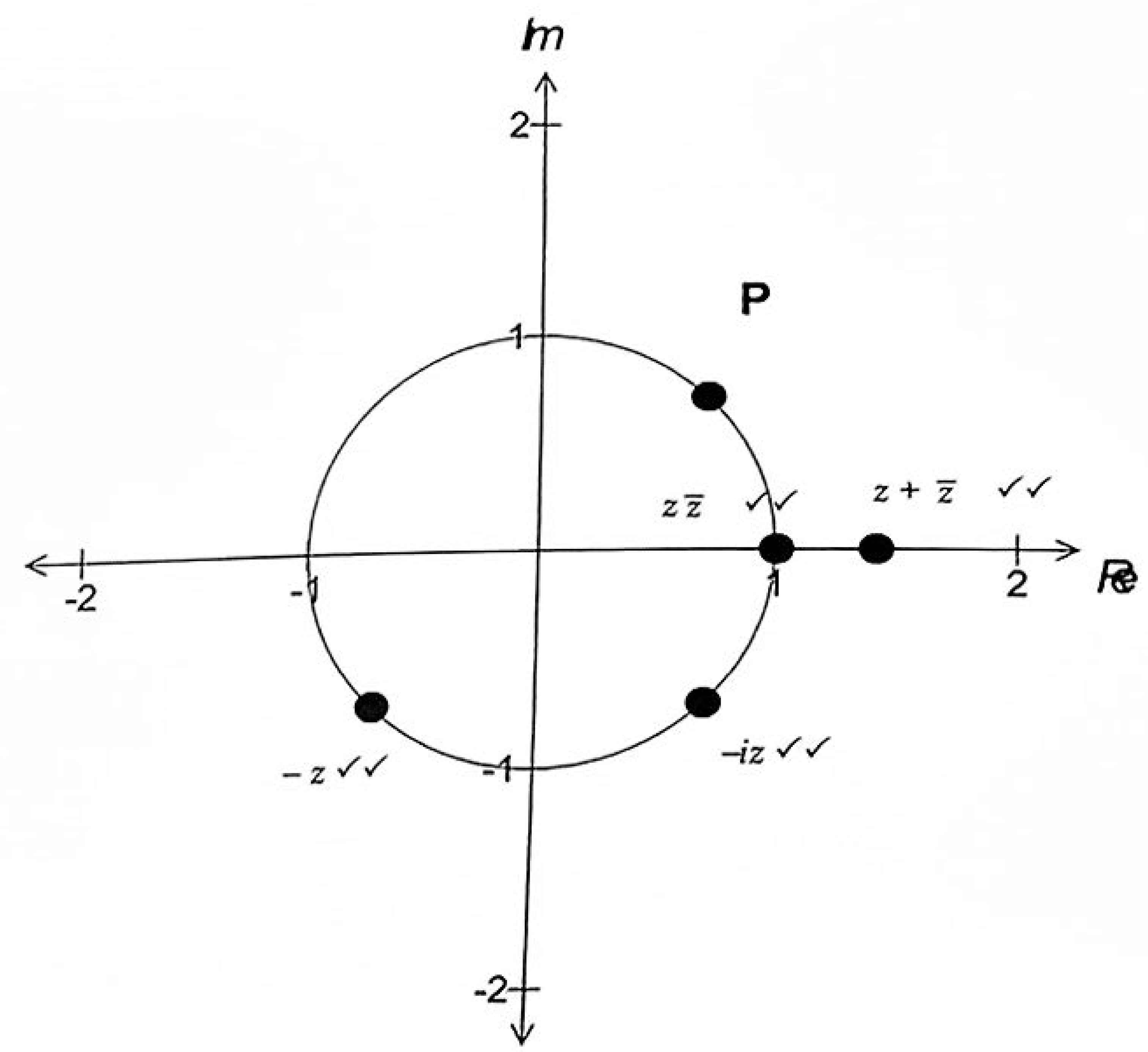
AB = $z_2 - z_1$ ✓✓

(ii) z is a complex number represented by the point Z such that $z_1 - z_2 = iz$. Mark on the Argand diagram above the position(s) of the point Z.

Calculator Free

8. [13 marks: 8, 5] [TISC]

(a) The complex number z where $|z| = 1$, is represented by the point P as marked in the Argand diagram below.

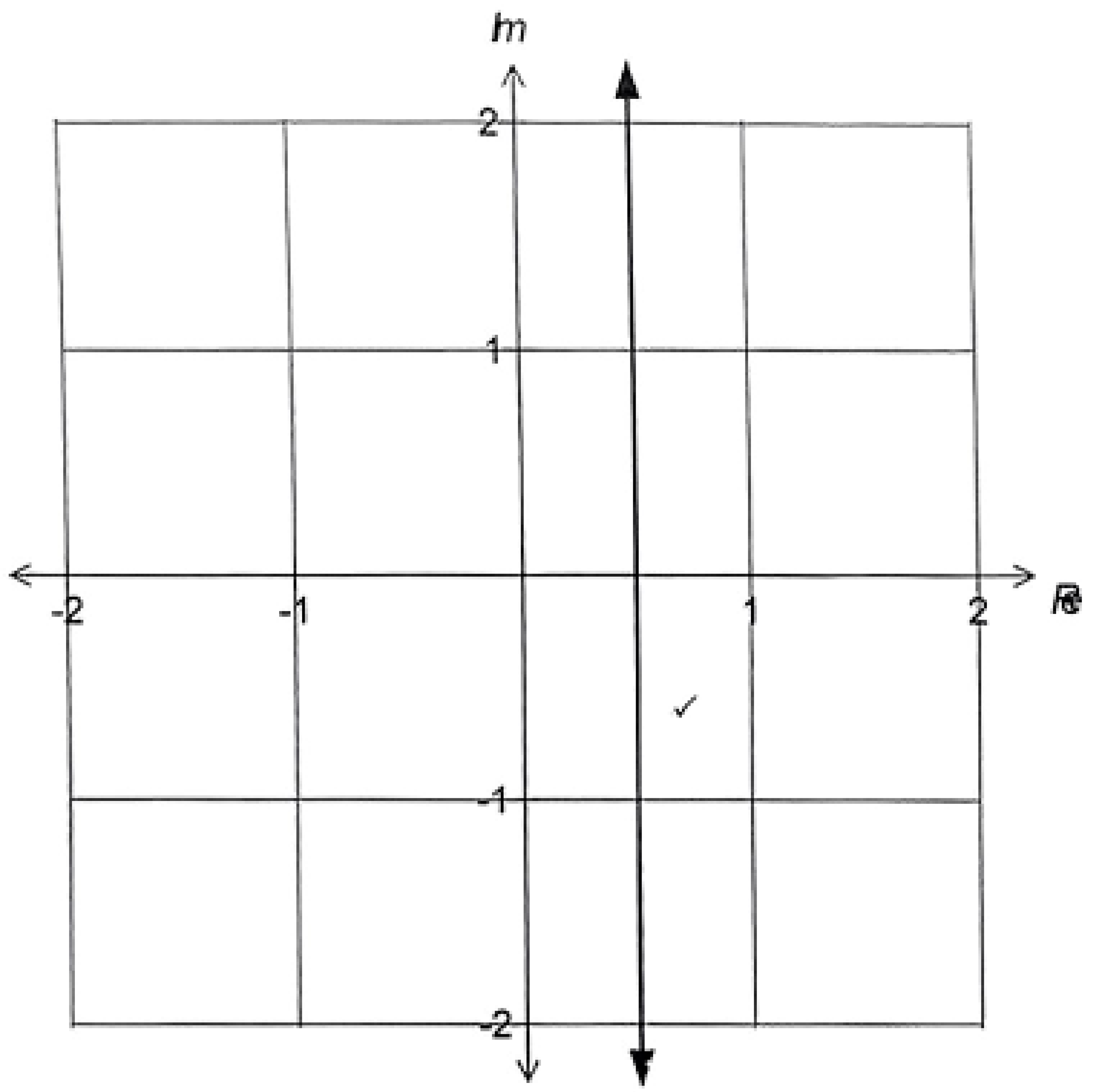


Mark clearly on the diagram above the points representing the complex numbers:

(i) $-z$ (ii) $-i\,z$ (iii) $z + \bar{z}$ (iv) $z \times \bar{z}$

Calculator Free

8. (b) The locus of the complex number z satisfies the equation $|z-1| = |\bar{z}|$.
Find the Cartesian equation of the locus and hence sketch the locus of z in the Argand diagram provided below.



Let $z = x + yi$.

$|(x-1) + yi| = |x - yi|$ ✓

$(x-1)^2 + y^2 = x^2 + y^2$ ✓

$x^2 - 2x + 1 = x^2$ ✓

$x = \frac{1}{2}$ ✓

Calculator Free

9. [7 marks: 3, 2, 2] [TISC]

The complex number z is defined by $z = \frac{a+4i}{i} + \frac{4}{1+i}$ where a is a real constant.

(a) Rewrite z in the form $x + y\,i$ where x and y are real.

$z = \frac{a+4i}{i} + \frac{4}{1+i}$

$= \frac{(a+4i)i}{i \times i} + \frac{4}{1+i} \times \frac{1-i}{1-i}$ ✓

$= \frac{-4+ai}{-1} + \frac{4-4i}{2}$ ✓

$= 6 - (a+2)i$ ✓

(b) Find the value of a if z lies on the line $Im(z) = -Re(z)$.

$-(a+2) = -6$ ✓

$a = 4$ ✓

(c) Show that z cannot lie on the curve $arg(z) = \frac{3\pi}{4}$.

If $\arg(z) = \frac{3\pi}{4}$, then $\text{Re}(z) \le 0$. ✓

but $\text{Re}(z) = 6 > 0$.

Hence, z cannot lie on $\arg(z) = \frac{3\pi}{4}$. ✓

Calculator Assumed

10. [8 marks: 3, 5] [TISC]

Let $w = x + yi$.

(a) If $\left|\dfrac{w}{1-w}\right| = 1$, show that w lies on the line with equation $x = \dfrac{1}{2}$.

$\left|\dfrac{w}{1-w}\right| = 1 \Rightarrow |w| = |1-w|$ ✓

$x^2 + y^2 = (1-x)^2 + y^2$ ✓

$x^2 + y^2 = 1 - 2x + x^2 + y^2$

$\Rightarrow \quad x = \dfrac{1}{2}$ ✓

(b) If $\left|\dfrac{w}{1-w}\right| = 3$, show that w lies on a circle. Find the equation of this circle.

$\left|\dfrac{w}{1-w}\right| = 3 \Rightarrow |w| = 3|1-w|$ ✓

$x^2 + y^2 = 9[(1-x)^2 + y^2]$ ✓

$8x^2 - 18x + 8y^2 = -9$

$(x - \dfrac{9}{8})^2 + y^2 = \dfrac{9}{64}$ ✓

This is the equation of a circle

centre $(\dfrac{9}{8}, 0)$ and radius $\dfrac{3}{8}$ ✓✓

Calculator Assumed

11. [8 marks]

The locus of the complex number z satisfies the equation $\left|\dfrac{z-1+2i}{z-1-2i}\right| = 2$.

Find the Cartesian equation of the locus. Hence sketch the locus of z.

$\left|\dfrac{z-1+2i}{z-1-2i}\right| = 2 \quad \Rightarrow |z-1+2i| = 2|z-1-2i|$ ✓

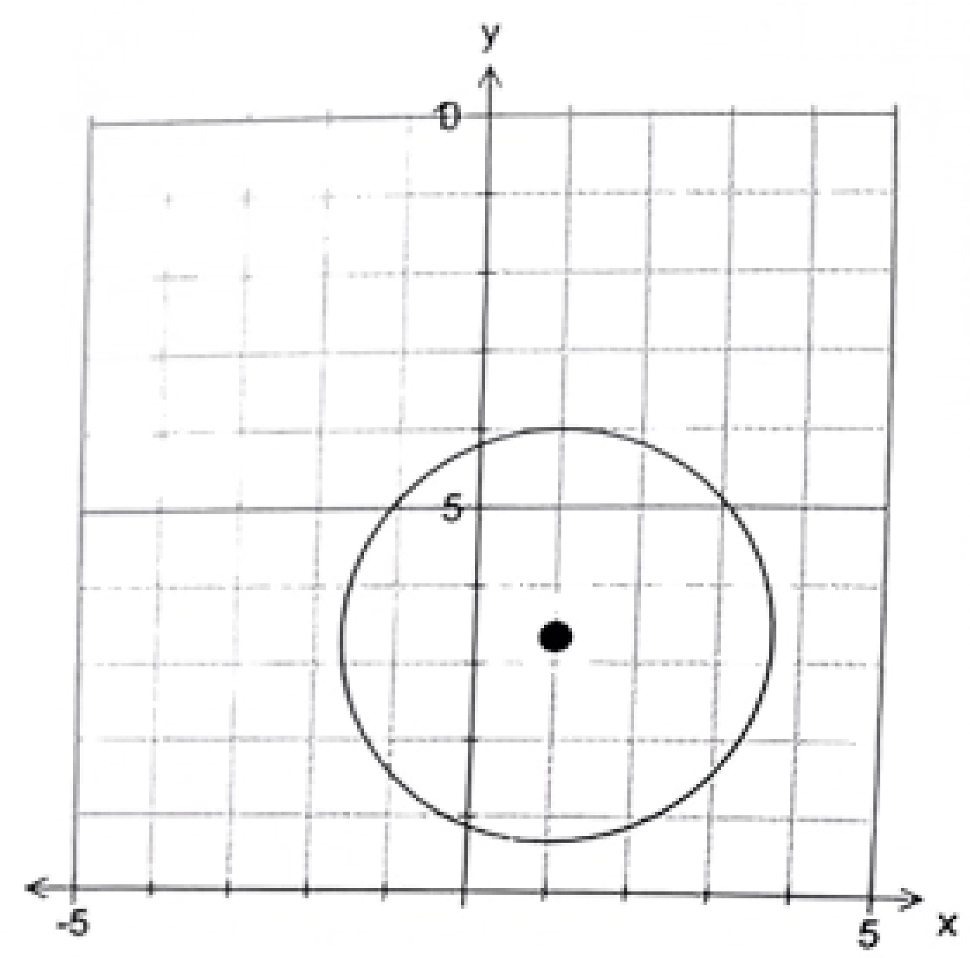
$(x-1)^2 + (y+2)^2 = 4[(x-1)^2 + (y-2)^2]$ ✓

$3x^2 - 6x + 3y^2 - 20y + 15 = 0$ ✓

$(x-1)^2 + (y - \dfrac{10}{3})^2 = \dfrac{64}{9}$ ✓

This is the equation of a circle

centre $(1, \dfrac{10}{3})$ and radius $\dfrac{8}{3}$. ✓✓



✓✓